Uniformly closed Fourier algebras

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At the end of his paper *The class of functions which are absolutely convergent Fourier transforms* [7], SEGAL remarks without proof that he and KAPLANSKY have proved the following theorem:

Theorem A. *If G is a locally compact group with the property that $L^1(G)$ is closed in the norm $\|f\| = \sup\{\|f*g\|_2 : g \in L^2(G)\}$, then G is finite.*

Theorem A is offered as a non-commutative version of the main result of Segal's paper which we state as follows:

Theorem B. *If G is a locally compact abelian group with the property that $L^1(G)$ is closed in the norm $\|f\| = \sup\{|\hat{f}(\chi)| : \chi \in \hat{G}\}$, then G is finite.*

The Plancherel theorem provides the perspective which assures us that for abelian G, the two norms are the same.

It is easy to see that the following is equivalent to Theorem A:

Theorem A′. *If the $*$-algebra $L^1(G)$ can be renormed as a C^*-algebra, G is finite.*

A different generalization of Theorem B to non-commutative groups is stimulated by recent advances in non-commutative harmonic analysis, especially the contributions of EYMARD [3] and WALTER [9]. In fact, since for abelian G, G is finite if and only if $\hat{G}$ is finite, we can rewrite Theorem B as follows:

Theorem B′. *If G is a locally compact abelian group such that $A(G) = \{\hat{f} : f \in L^1(\hat{G})\}$ is uniformly closed, G is finite.*

For a general locally compact group, $B(G)$, the Fourier—Stieltjes algebra of G, is the complex linear span of the continuous, positive definite functions on G.

*) Research supported in part by the National Research Council of Canada.

Normed as the dual of $C^*(G)$, and with pointwise algebraic operations, $B(G)$ is a commutative Banach algebra with identity, closed with respect to complex conjugation. $A(G)$, the Fourier algebra of G, is the closed ideal in $B(G)$ composed of precisely those functions of the form $f * g^{\sim}$, $f, g \in L^2(G)$. It is far from clear that this set of functions should have any of the algebraic or topological properties here ascribed to it. But see EYMARD [3].

At any rate, if G is abelian,

$$A(G) = \{\hat{f} : f \in L^1(\hat{G})\} \quad \text{and} \quad B(G) = \{\hat{\mu} : \ \mu \in M(\hat{G})\},$$

with the norms transported from $L^1(\hat{G})$, $M(\hat{G})$ respectively, which accounts for the names introduced above, and we can ask for the extension of Theorem B′ to non-abelian G:

Theorem C. *If $A(G)$ is uniformly closed, G is finite.*

A consequence of Theorem C would be

Theorem D. *If $B(G)$ is uniformly closed, G is finite.*

For, by uniqueness of complete norm topology, since A is closed in B, A would be uniformly closed.

Analogously, one could rewrite Theorem A (or A′) with $M(G)$ in place of $L^1(G)$.

We will prove all of these theorems, and prove their equivalence. More precisely, we will prove

Main Theorem. *Let G be a locally compact Hausdorff topological group. The following are equivalent*

i) *$A(G)$ is uniformly closed.*
ii) *The $*$-algebra $A(G)$ can be renormed as a C^*-algebra.*
iii) *$B(G)$ is uniformly closed.*
iv) *The $*$-algebra $B(G)$ can be renormed as a C^*-algebra.*
v) *$L^1(G)$ is uniformly closed in the left regular representation.*
vi) *The $*$-algebra $L^1(G)$ can be renormed as a C^*-algebra.*
vii) *$M(G)$ is uniformly closed in the left regular representation.*
viii) *The $*$-algebra $M(G)$ can be renormed as a C^*-algebra.*
ix) *The $*$-algebra $M(G)$ can be renormed as a W^*-algebra.*
x) *G is finite.*

(In references to $A(G)$ or $B(G)$ as $*$-algebras, complex conjugation is to be used as involution.)

Our interest in the question arose from WALTER's work [9], which shows $A(G)$ {respectively $B(G)$} to be a complete Banach-algebra invariant of the locally compact group G. This leads us to expect much more of the commutative picture to persist into the non-commutative setting than previously seemed reasonable.

Along the way we prove another result of interest: a locally compact group which is extremely disconnected must be discrete.

All the topological groups considered are assumed Hausdorff.

§ 1. Preliminaries

The following well-known lemma was first explicitely stated and proved in [8], for which reference I thank the referee. Professor I. HALPERIN has kindly pointed out that it can be proved as an easy consequence of [5], Theorem 1. We include a short proof.

Lemma 1. *A norm-separable W^*-algebra is finite dimensional.*

Proof. Let A be such an algebra. If A contains infinitely many mutually orthogonal projections, A contains a copy of l^∞ which is not norm separable. Therefore A has no type II or type III part, and A is a finite sum of n-homogeneous type I summands, each of the form $D\otimes L(H)$, with D commutative [1]. But since D (respectively $L(H)$ can have only finitely many mutually orthogonal projections, D and $L(H)$ must be finite dimensional. The lemma follows.

Lemma 2. *An extremely disconnected compact topological group is finite.*

Proof. Since an extremely disconnected space (one in which the closure of each open set is open) is totally disconnected, we have at once that the underlying space of G is $\{0, 1\}^m$, where m is a suitable cardinal number [4], Theorem 9. 15. We need only show that such a space is extremely disconnected only if it is finite.

First, assume $m=\aleph_0$. Then $X=\{0, 1\}^m$ is metric, with $d(x,0)=\sum_{i=1}^{\infty} x_i 2^{-i}$. Especially, $x\to d(x, 0)$ maps X onto $[0, 1]$.

Let $r\in[0, 1]$ be not a dyadic rational, and consider the open r-ball B about 0 in X. Since the dyadic representation of r is unique, and there exists a point $p\in X$ at distance r from 0, p is in the closure $\bar{B}$ of B, while no ball about p is in $\bar{B}$. Thus $\bar{B}$ is not open.

Now for arbitrary infinite cardinal m, keeping the notation X, B as above, we can write $\{0, 1\}^m = X\times\{0, 1\}^d$. Then $U = B\times\{0, 1\}^d$ is open, and $\bar{U} = \bar{B}\times\{0, 1\}^d$ cannot be open, or its projection $\bar{B}$ would be open in X. This concludes the proof.

Now we generalize to non-compact groups.

Definition. A topological space is *locally extremely disconnected* (L.E.D.) if every open subset with compact closure has open closure.

Lemma 3. *An open subspace A of L.E.D. Hausdorff space X is L.E.D.*

Proof. If A is open in X, and U is open in A with $\overline{U}\cap A$ compact, then U is open in X, and $\overline{U}\cap A$ is closed, so $\overline{U} = \overline{U}\cap A$ is compact, so open in X, so in A.

Proposition 4. *A locally extremely disconnected, locally compact topological group is discrete.*

Proof. Let V be a relatively compact, open neighborhood of e in G. Then $\overline{V}$ is a compact open neighborhood of e, and contains a compact, open subgroup H [4], Theorem 7. 5. Then H is an extremely disconnected compact group, so by Lemma 2, H is finite. Since H is open, G is discrete.

Corollary 5. *An extremely disconnected locally compact topological group is discrete.*

§ 2. Proof of the main theorem

Terminology. The left regular representation of a locally compact group G {or of $L^1(G)$, or of $M(G)$} is the restriction to G {or to $L^1(G)$, or to $M(G)$} of the action of M on $L^2(G)$ by left convolution: If $L_\mu(g) = \mu * g$, $(g\in L^2, \mu\in M)$ $\mu\to L_\mu$ is the left regular representation.

The weakly closed algebra generated by the image of L (this is unambiguous) is the left regular von Neumann algebra of G.

To avoid artificiality, we prove more than a minimal chain of implications. From those we prove the schemes

$$\begin{array}{ccc} \text{ii)} & & \\ \Updownarrow & & \\ \text{i)} \Leftrightarrow \text{iii)} \Rightarrow \text{x)}, & \text{and} & \text{ix)} \Leftrightarrow \text{viii)} \Rightarrow \text{vi)} \Rightarrow \text{iii)} \Rightarrow \text{x)} \end{array}$$

with vii) $\Updownarrow$ above viii) and v) $\Updownarrow$ above vi),

are easily extracted, which together with the fact that x) implies all the others serve to establish the equivalence.

The following implications are trivial: i)$\Rightarrow$ii); iii)$\Rightarrow$iv); v)$\Rightarrow$vi), vii)$\Rightarrow$viii), ix)$\Rightarrow$viii), x)$\Rightarrow$all others.

The implications ii)$\Rightarrow$i), iv)$\Rightarrow$iii), vi)$\Rightarrow$v), vii)$\Rightarrow$vii) all follow from the fact that the image of a C^*-algebra under any $*$-representation on a Hilbert space is again a C^*-algebra. For the first two implications, take as the representation $f\mapsto M_f\in L(L^2(G))$, where $M_f(g)=fg$ $(g\in L^2)$.

vi)$\Rightarrow$iii): If the $*$-algebra L^1 can be renormed as a C^*-algebra, the new norm $\|\cdot\|$ is smaller than the $\|\cdot\|_1$-norm, since the identity map is norm-reducing on $(L^1, \|\ \|_1)$ to $(L^1, \|\ \|)$. Therefore the two norms are equivalent. L^∞ thus becomes, with a change of norm, but not of norm topology, the dual space of a C^*-algebra, and hence the linear span of the positive linear functionals. But these can be chosen in L^∞ to be continuous positive definite functions. So L^∞ consists entirely of continuous functions, $L^\infty = B$, and B is uniformly closed.

The implications iii)⇒i), iv)⇒ii), vii)⇒v), viii)⇒vi) follow immediately from the fact that A is closed in B (respectively L^1 is closed in M), together with the equivalence of norms proved in the preceding paragraph.

vii)⇒ix) follows from SAKAI's characterization [6] of W^*-algebras as C^*-algebras which are dual spaces, since M is the dual of C_0. Again we use the equivalence of the original norm and the C^*-norm on M.

i)⇒ix): Since $A(G)$ is dense in $C_0(G)$ [3], i) is equivalent to $A(G)=C_0(G)$ as a topological vector space. But Eymard has shown that A^d=the left regular von Neumann algebra of G, while $C_0^d=M$. This gives us ix).

iii)⇒x): We have established that i)⇒ix)⇒viii)⇒vi)⇒iii)⇒i), so that iii)⇒vi). Therefore, if G satisfies iii), $L^\infty=B$, as shown in the proof that vi)⇒iii). Since L^∞ is a W^*-algebra, its spectrum is its $*$-spectrum (the set of $*$-homomorphisms onto $\mathbf{C}$, with the w^*-topology), and is an extremely disconnected compact Hausdorff space. Thus the $*$-spectrum Δ of B is extremely disconnected. But Δ is a compactification of G, and G, being locally compact, is open in every compactification. Thus by Lemma 3, G is locally extremely disconnected, and finally, by Proposition 4, G is discrete.

Now supposing G is infinite, we derive a contradiction. In fact, suppose H is a denumerably infinite subgroup of G. Then $M(H)=L^1(H)$ is a closed subalgebra of $M(G)=L^1(G)$, and so (iii)⇒vi)) can be renormed as a C^*-algebra, which is then a W^*-algebra (see viii)⇒ix)).

But $L^1(H)$ is norm separable, so by Lemma 1 finite dimensional, making H finite. This contradiction completes the proof.

In [2], R. E. EDWARDS proved that if G is a locally compact abelian group, and A a commutative C^*-algebra, $\varphi\colon A\to M(G)$ an algebraic isomorphism into $M(G)$, then A is finite dimensional.

We remark finally that the following theorem, which is a non-commutative version of Edwards' theorem, can be used to secure the implication vi)⇒x), replacing the argument involving extremely disconnected groups.

Theorem 6. *If A is a closed $*$-subalgebra of $M(G)$ which can be renormed as a C^*-algebra, then A is finite dimensional.*

The proof is an easy consequence of Ogasawara's Theorem 1, op. cit., together with Edvards' arguments. We omit the detail.

Problem. Is Theorem 6 valid with $B(G)$ in place of $M(G)$?

Added in proof. The problem is solved affirmatively by C. A. AKEMANN in private correspondence. Akemann alerts us to Sakai's theorem (*Proc. Japan Acad.*, **33** (1957), 439—444) asserting that the predaul of a W^*-algebra is sequentially weakly complete. Then the argument sketched above for $M(G)$ applies.

We thus obtain the stronger form of the main theorem:

If A is a subalgebra of $B(G)$ (a $*$-subalgebra of $M(G)$ which is ($*$-) isomorphic to a C^*-algebra, then A is finite dimensional.

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(Received April 29, 1971)