On a generalization of Bernoulli's inequality

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At the Oberwolfach Conference on "Linear Operators and Approximation" in August, 1971, H. S. SHAPIRO [1] presented a lecture on "Fourier multipliers whose multiplier norm is an attained value". On this occasion he mentioned that the elementary inequalities

$$|1+z|^4 \geqq 1+4\operatorname{Re} z+|z|^2+\frac{1}{5}|z|^4, \tag{1}$$

$$|1+z|^4 \leqq 1+4\operatorname{Re} z+8|z|^2+3|z|^4 \tag{2}$$

played an important role in the proof of his Theorem 2. (See Lemma 5 and 6 in [1].) He also stated (see also [2]) that to prove an analogue of his result for $p\geqq 2$ inequalities of the following type are needed:

$$|1+z|^p \geqq 1+p\operatorname{Re} z+a_p|z|^2+b_p|z|^p, \tag{3}$$

$$|1+z|^p \leqq 1+p\operatorname{Re} z+A_p|z|^2+B_p|z|^p, \tag{4}$$

where a_p, b_p, A_p, B_p are positive constants depending only on p.

The proof of (3) and (4) given in [2] does not seem to yield optimal values for these positive constants. In connection with this fact H. S. SHAPIRO raised the problem to find the best possible constants, i.e. the exact range of (a_p, b_p) such that (3) holds, and similarly for (4).

In the present paper we are going to give a proof of these inequalities which exhibits best possible constants. In fact we prove:

Theorem. *For any complex number z and for any $p\geqq 2$ the inequalities*

$$|1+z|^p \geqq 1+p\operatorname{Re} z+a_p|z|^2+b_p|z|^p, \tag{5}$$

$$|1+z|^p \leqq 1+p\operatorname{Re} z+A_p|z|^2+B_p|z|^p \tag{6}$$

hold with any positive a_p, b_p, A_p, B_p satisfying

$$0 < a_p < \frac{p}{2}, \tag{7}$$

(8) $$0 < b_p \leqq \mu_1(p) = \min_{t \geqq 2} \frac{(t-1)^p + pt - 1 - a_p t^2}{t^p},$$

(9) $$1 < B_p < \infty,$$

(10) $$M_1(p) = \sup_{t>0} \frac{(t+1)^p - 1 - pt - B_p t^p}{t^2} \leqq A_p < \infty,$$

or

(7′) $$0 < b_p < \mu_2(p) = \min_{t \geqq 2} \frac{(t-1)^p + pt - 1}{t^p},$$

(8′) $$0 < a_p \leqq \mu_3(p) = \min_{t \geqq 2} \frac{(t-1)^p + pt - 1 - b_p t^p}{t^2},$$

(9′) $$\frac{p(p-1)}{2} < A_p < \infty,$$

(10′) $$M_2(p) = \sup_{t>0} \frac{(t+1)^p - 1 - pt - A_p t^2}{t^p} \leqq B_p < \infty.$$

These ranges of (a_p, b_p) and (A_p, B_p) are best possible.

Remarks. For some special p the exact values or good approximation of the numbers μ_i and M_i can be gotten by an easy computation.

For instance if $p=2$ then $\mu_1(2) = 1-a_2$, $M_1(2) = 1-B_2$,

$$\mu_2(2) = 1, \quad \mu_3(2) = 1-b_2 \quad \text{and} \quad M_2(2) = 1-A_2.$$

If $p=4$ and we choose $a_4=1$ then $\mu_1(4)=\frac{1}{5}$, and, for $B_4=3$, $M_1(4)=8$; i.e. the constants $\frac{1}{5}$ and 8 appearing in (1) and (2) are optimal.

The following estimates of the numbers μ_i and M_i can be obtained by a standard computation:

$$\mu_1(p) \leqq \frac{2p-4a_p}{2^p}; \quad \mu_2(p) \leqq \frac{2p}{2^p}; \quad \mu_3(p) \leqq \frac{2p-2^p b_p}{4};$$

$$M_1(p) \geqq \max\left[2^p-1-p-B_p, \ \frac{p(p-1)}{2}\right]; \quad M_2(p) \geqq 2^p-1-p-A_p;$$

$$\mu_1(p) \geqq \frac{p-2a_p}{2}\left(\frac{a_p}{p}\right)^{p-2}\left(1-\frac{1}{p-1}\right)^{p-1};$$

$$\mu_2(p) \geqq \frac{1}{2^p} \quad \text{and} \quad \mu_3(p) \geqq \frac{1-2^p b_p}{4}.$$

One more remark: inequality (5) can be slightly generalized as follows.

For any $p \geqq q \geqq 2$ *the inequality*

$$|1+z|^p \geqq 1+p \operatorname{Re} z + a_p(q)|z|^q + b_p(q)|z|^p \tag{11}$$

holds, where

$$0 < a_p(q) < \min_{t \geqq 2} \frac{(t-1)^p + pt - 1}{t^q},$$

$$0 < b_p(q) \leqq \min_{t \geqq 2} \frac{(t-1)^p + pt - 1 - a_p(q) t^q}{t^p}.$$

The proof of these inequalities is the same as that of (7) and (8).

Such a generalization of (6) is impossible. This fact can be seen easily if p and q are integers greater than two and z is a real number tending to zero.

Proof of (5). Denote $z = x+iy$ and $r=|z|$. For the sake of brevity we write a for a_p and b for b_p.

In the first step we fix p, q, and r. Then we have to prove the inequality

$$(1+2x+r^2)^{\frac{p}{2}} \geqq 1+px+ar^2+br^p \tag{12}$$

for all x lying in $[-r, r]$ with positive a and b. Put $R = \frac{1}{2}(1+r^2)$, $C = 1+ar^2+br^p$,

$$f(x) = 2^{\frac{p}{2}}(R+x)^{\frac{p}{2}} \quad \text{and} \quad g(x) = px+C.$$

Drawing the graphs of these functions it is easy to see that inequality (12) will be satisfied if the graph of $y=g(x)$ lies under the graph of $y=f(x)$ on the interval $[-r, r]$. We obtain the best possible result in respect to a and b if $y=g(x)$ is tangent to the graph of $y=f(x)$ inside of $[-r, r]$ or, when this is not the case, if $y=g(x)$ passes through the point $P(-r, f(-r))$. (See Fig. 1 and Fig. 2.)

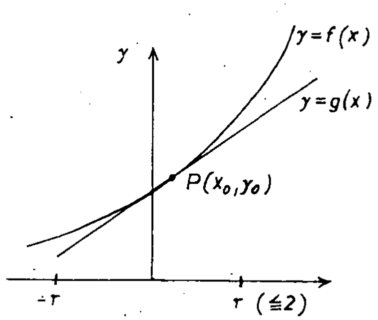

Fig. 1

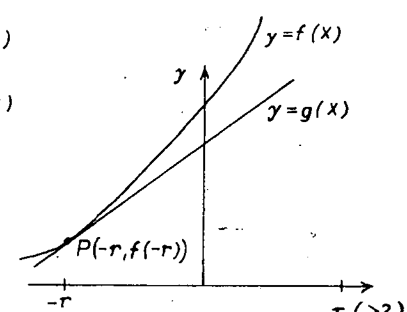

Fig. 2

In the first case our task reduces to find the point $P(x_0, y_0)$ on the graph $f(x)$ at which $f'(x_0)=p$,

In order to find $P(x_0, y_0)$ we calculate the derivative of $f(x)$ and solve the equation

$$f'(x) = p2^{\frac{p}{2}-1}(R+x)^{\frac{p}{2}-1} = p.$$

Thus we get $x_0 = -\frac{r^2}{2}$ and this x_0 lies in $[-r, r]$ if $r \leqq 2$.

In this case, i.e. if $r \leqq 2$, we obtain the best possible constants a and b if $f(x_0)=$ $=g(x_0)$, i.e. if

$$f\left(-\frac{r^2}{2}\right) = 1 = p\left(-\frac{r^2}{2}\right)+1+ar^2+br^p = g\left(-\frac{r^2}{2}\right)$$

holds. Hence we obtain the following conditions on a and b:

$$0 < a < \frac{p}{2} \quad \text{and} \quad 0 < b \leqq \frac{p-2a}{2^{p-1}}.$$

If $r \geqq 2$ we have the following equation as a condition on a and b:

$$f(-r) = (r-1)^p = p(-r)+1+ar^2+br^p = g(-r).$$

Hence we get

$$0 < a < u_1 = \min_{r \geqq 2} u_1(r) \equiv \min_{r \geqq 2} \frac{(r-1)^p+pr-1}{r^2} \leqq \frac{p}{2}$$

and

$$0 < b \leqq u_2 = \min_{r \geqq 2} u_2(r) \equiv \min_{r \geqq 2} \frac{(r-1)^p+pr-1-ar^2}{r^p} \leqq \frac{p-2a}{2^{p-1}}.$$

It is easy to see that u_1 and u_2 are positive. In fact, $u_1(r) \geqq \frac{p}{r}$ for any $r \geqq 2$ and

$$u_1(r) \to \begin{cases} \infty & \text{for} \quad p>2 \\ 1 & \text{for} \quad p=2 \end{cases} \quad \text{as} \quad r \to \infty.$$

Similarly, $u_2(r) \geqq \frac{u_1 r^2 - ar^2}{r^p} = \frac{u_1 - a}{r^{p-2}} > 0$ for any $r \geqq 2$; furthermore

$$u_2(r) \to \begin{cases} 1 & \text{for} \quad p > 2 \\ 1-a & \text{for} \quad p = 2 \end{cases} \quad \text{as} \quad r \to \infty.$$

To complete the proof of (5) we have only to show that $u_1 = \frac{p}{2}$. To verify this we compute

$$\big(u_1(r)\big)' = r^{-4}\big[(p(r-1)^{p-1}+p)r^2-2r((r-1)^p+pr-1)\big].$$

Let $h(r) = \big(p(r-1)^{p-1}+p\big)r-2\big((r-1)^p+pr-1\big)$. Now $h(2)=0$ and

$$h'(r) = p\big[(p-2)(r-1)^{p-1}+(p-1)(r-1)^{p-1}-1\big] \geqq 0$$

for all $r\geqq 2$; thus $h(r)\geqq 0$, which implies that $u_1'(r)\geqq 0$, i.e. $u_1(r)$ is an increasing function, hence $u_1 = \min_{r\geqq 2} u_1(r) = u_1(2) = \frac{p}{2}$ in accordance with our statement.

Setting $\mu_1(p)=u_2$ and collecting our results the proof of (5) is complete.

Proof of (6). We use the same notations as before except that we write a for A_p and b for B_p. We distinguish the cases $0\leqq r\leqq 1$ and $r>1$. If $r\leqq 1$ then let $h_1(r) = (r+1)^p-(1-r)^p-2rp$. Since $h_1(0)=0$ and $h_1'(0)=0$, furthermore $h_1''(r)\geqq 0$ for all $0\leqq r\leqq 1$, we have

(13) $$h_1(r)\geqq 0 \quad \text{for all} \quad 0\leqq r\leqq 1.$$

If $r\geqq 1$ let $h_2(r) = (r+1)^p-(r-1)^p-2rp$. As before, since $h_2(1)\geqq 0$, $h_2'(1)\geqq 0$ and $h_2''(r)\geqq 0$ for all $r\geqq 1$, we have

(14) $$h_2(r)\geqq 0 \quad \text{for all} \quad r\geqq 1.$$

(13) and (14) imply that

(15) $$\frac{f(r)-f(-r)}{2r} = \frac{(r+1)^p-|r-1|^p}{2r} \geqq p$$

for all $r>0$.

By (15) it is evident that inequality (6) is satisfied if $g(r)\geqq f(r)$ and we obtain the optimal constants if $g(r)=f(r)$.

Hence we get the following condition on a and b:

$$1+pr+ar^2+br^p = (r+1)^p.$$

It is easy to see that b must be greater than 1, and if b is fixed then the best possible value of a is

$$a = \sup_{r>0} \frac{(r+1)^p-1-pr-br^p}{r^2}.$$

To complete the proof of (6) we have only to prove that $\sup_{r>0} v(r)<\infty$, where

$$v(r) = \frac{(r+1)^p-1-pr-br^p}{r^2}.$$

In fact, since $b > 1$,

$$v(r) \to \begin{cases} -\infty & \text{for} \quad p > 2 \\ 1-b & \text{for} \quad p = 2 \end{cases} \quad \text{as} \quad r \to \infty$$

and

$$v(r) \to \begin{cases} \dfrac{p(p-1)}{2} & \text{for} \quad p > 2 \\ 1-b & \text{for} \quad p = 2 \end{cases} \quad \text{as} \quad r \to 0,$$

and these statements imply the desired conclusion.

The proof of (6) is complete.

Inequalities (5) and (6) with (7′), (8′), (9′) and (10′) can be proved similarly, and therefore these proofs are omitted.

References

[1] SHAPIRO, H. S. Fourier multipliers whose multiplier norm is an attained value, *Linear Operators and Approximation* (Proceedings of the Conference in Oberwolfach, 1971), to appear.

[2] FEFFERMAN, C. and H. S. SHAPIRO, A planar face on the unit sphere of the multiplier space M_p, $1 < p < \infty$, *Proc. Amer. Math. Soc.* (to appear).

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