The non-orthogonal Menchoff—Rademacher theorem

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1. Introduction. A well-known result of RADEMACHER [1] and MENCHOFF [2] states that if $\varphi_1, \varphi_2, \ldots, \varphi_N$ is a finite set of orthogonal functions in some L_2 space of finite measure, then there exists a $\delta \in L_2$ such that

$$\left|\sum_{k=1}^{n} \varphi_k(x)\right| \leqq \delta(x) \quad \text{for} \quad 1 \leqq n \leqq N$$

and

$$\|\delta\| \leqq \log_2 4N \left(\sum_{k=1}^{N} \|\varphi_k\|^2\right)^{1/2}. \tag{1.1}$$

This result implies the convergence a.e. of any orthogonal series $\sum_{k=1}^{\infty} \varphi_k(x)$ satisfying

$$\sum_{k=1}^{\infty} \|\varphi_k\|^2 \log^2 k < +\infty. \tag{1.2}$$

According to results of MENCHOFF [2] and TANDORI [3], condition (1.1) cannot be weakened in general, i.e. the factor $\log 4N$ cannot be replaced by $o(\log N)$. I. S. GÁL [4], [5], [6] and more recently R. I. SERFLING [7], [8] have generalized the above statements for spaces L_p with $p \geqq 2$ and for non-orthogonal series. In the present paper we shall give a new, simpler proof of Serfling's theorem, published in [7].

The method of proof is similar to that of Rademacher and Menchoff. In Ch. 2 we shall derive some remarkable consequences of Serfling's theorem.

2. Let L_p be a Lebesgue space $(2 \leqq p < \infty)$ on some measure space with integral $M(.)$. Let $g(i, j)$ be a non-negative function defined for integers i, j $(1 \leqq i \leqq j \leqq N)$ such that

$$g(i, j) + g(j+1, k) \leqq g(i, k) \quad \text{for} \quad 1 \leqq i \leqq j < k \leqq N. \tag{2.1}$$

Furthermore, for any $\varphi_i \in L_p$ $(1 \leqq i \leqq N)$, set

$$s_{ij}(x) = \sum_{i<k\leqq j} \varphi_k(x) \quad \text{for} \quad 0 \leqq i < j \leqq N$$

and

$$s_j(x) = s_{0j}(x) \quad \text{for} \quad 1 \leqq j \leqq N.$$

Theorem 1. *If*

$$\|s_{ij}\|_p \leqq g(i+1, j) \quad \text{for} \quad 0 \leqq i < j \leqq N \tag{2.2}$$

then there exists a function $\delta \in L_p$ such that

$$|s_j(x)| \leqq \delta(x) \quad \textit{for} \quad 1 \leqq j \leqq N \tag{2.3}$$

and

$$\|\delta\|_p \leqq (\log_2 4N)^2 g(1, N). \tag{2.4}$$

Proof. Suppose first that $N=2^r$ with some integer $r>0$ and define

$$q_k^{(j)} = s_{2^{r-k}j,\, 2^{r-k}(j+1)} \quad \text{for} \quad 0 \leqq j \leqq 2^{k-1},\ 0 \leqq k \leqq r$$

and

$$Q_k = \sum_{j=0}^{2^k-1} |q_k^{(j)}|^p \quad \text{for} \quad 0 \leqq k \leqq r.$$

Every n satisfying $1 \leqq n \leqq N$ can be uniquely represented in the form

$$n = \alpha_0 + \alpha_1 \cdot 2 + \cdots + \alpha_r \cdot 2^r,$$

where $\alpha_l = 0$ or 1 for $0 \leqq l \leqq r$. Accordingly we shall have

$$s_n = \sum_{k=0}^{r} \alpha_k q_k,$$

where $q_k = q_k^{(j)}$ for suitable $j = j(n, k)$. Applying Hölder's inequality we get

$$|s_n(x)| \leqq \sum_{k=0}^{r} \alpha_k |q_k(x)| \leqq \left(\sum_{k=0}^{r} |q_k(x)|^p \right)^{1/p} \left(\sum_{k=0}^{r} 1 \right)^{1/q}$$

where $1/p + 1/q = 1$. Clearly

$$Q_k(x) \geqq |q_k(x)|^p \qquad \text{for} \quad 1 \leqq k \leqq r,$$

hence

$$|s_n(x)|^p \leqq (r+1)^{p-1} \sum_{k=0}^{r} Q_k(x).$$

Obviously $\delta(x) = (r+1)^{1/q} \left(\sum_{k=0}^{r} Q_k(x) \right)^{1/p}$ statisfies condition 2.4 and $\delta \in L_p$.

Furthermore, we have

$$M(\delta^p) \leqq (r+1)^{p-1} \sum_{k=0}^{r} M(Q_k) = (r+1)^{p-1} \sum_{k=0}^{r} \sum_{j=0}^{2^k-1} M(|q_k^{(j)}|^p) \leqq$$

$$\leqq (r+1)^{p-1} \sum_{k=0}^{r} \sum_{j=0}^{2^k-1} \left[g\left(2^{r-k}j+1,\, 2^{r-k}(j+1)\right) \right]^{p/2}.$$

Here, in the last step, we used (2. 2). Since $p/2 \geqq 1$, Jensen's inequality yields:

$$\sum_{j=0}^{2^k-1} [g(2^{r-k}j+1, 2^{r-k}(j+1))]^{p/2} \leqq \left[\sum_{j=0}^{2^k-1} g(2^{r-k}j+1, 2^{r-k}(j+1))\right]^{p/2} \leqq [g(1, N)]^{p/2},$$

hence

$$M(\delta^p) \leqq (r+1)^p [g(1, N)]^{p/2},$$

i.e.

$$\|\delta\|^2 \leqq (r+1)^2 \cdot g(1, N).$$

As $r+1 \leqq \log_2 N+1 = \log_2 2N$ we have

$$\|\delta\|^2 \leqq (\log_2 2N)^2 \cdot g(1, N)$$

Let now N be arbitrary and denote by N' the first 2-power exceeding N. Define $\varphi_i \equiv 0$ for $N < i \leqq N'$, and

$$\bar{g}(i, j) = \begin{cases} g(i, j) & \text{if } 1 \leqq i \leqq j \leqq N, \\ g(i, N) & \text{if } 1 \leqq i \leqq N \leqq j \leqq N', \\ 0 & \text{if } N < i \leqq j \leqq N'. \end{cases}$$

The extended system $\{\varphi_i\}$ obviously satisfies the conditions of the theorem for $\bar{g}$; hence there exists a $\delta \in L_p$ such that, in particular,

$$|s_n(x)| \leqq \delta(x) \quad \textit{for} \quad 1 \leqq n \leqq N$$

and $\|\delta\|^2 \leqq (\log_2 2N')^2 \bar{g}(1, N')$. As $N' < 2N$, we have $\log_2 2N' \leqq \log_2 4N$ which completes the proof.

Corollary. *If $\varphi_1, \ldots, \varphi_N$ are arbitrary elements of an abstract L_2-space then there exists a $\delta \in L_2$ such that*

$$\left|\sum_{k=1}^{n} \varphi_k(x)\right| \leqq \delta(x) \quad \textit{for} \quad 1 \leqq n \leqq N$$

and

$$\|\delta\|^2 \leqq (\log_2 4N)^2 \cdot \sum_{i,j=1}^{N} |(\varphi_i, \varphi_j)|.$$

(Here (,) denotes the scalar product in L_2.)

Proof. Apply Theorem 1 with

$$g(i, j) = \sum_{k=i}^{j} \sum_{l=1}^{j} |(\varphi_k, \varphi_l)|.$$

3. Non-orthogonal series. In the sequel we shall deal with L_2 spaces only.

Theorem 2. *Let $\{\varphi_i\}_{i=1}^{\infty}$ be a sequence of elements of L_2, $\{\alpha_i\}_{i=1}^{\infty}$ a nondecreasing sequence of positive real numbers with $\alpha_i \to \infty$ as $i \to \infty$, and suppose that*

$$\sum_{i,j=1}^{\infty} |(\varphi_i, \varphi_j)| \alpha_i \alpha_j < +\infty. \tag{3. 1}$$

If $n(k)$ denotes the first integer m for which $k \leqq \alpha_m^2$, then $s_{n(k)} = \sum_{k=1}^{n(k)} \varphi_l$ converges a.e.

Proof. Obviously condition (3. 1) implies the convergence of the series $\sum_{k=1}^{\infty} \varphi_k$ in L_2.

Denote by f the sum of this series and define

$$d_n = \left| f - \sum_{k=1}^{n} \varphi_k \right|^2 = \sum_{i,j=n+1} (\varphi_i, \varphi_j).$$

If N is an arbitrary integer, $N > 1$, then

$$\sum_{k=1}^{N} d_{n(k)} = \sum_{k=1}^{N} [k-(k-1)] d_{n(k)} = \sum_{k=1}^{N-1} k(d_{n(k)} - d_{n(k-1)}) + N \cdot d_{n(N)} =$$

$$= \sum_{k=1}^{N-1} k \sum_{(n(k), n(k+1)]} (\varphi_i, \varphi_j) + N d_{n(N)};$$

here $(n(k), n(k+1)]$ denotes the set of all pairs (i, j) of integers for which $n(k) < \min(i, j) \leqq n(k+1)$. From the definition of α_i and $n(k)$:

$$|k \sum_{(n(k), n(k+1)]} (\varphi_i, \varphi_j)| \leqq \alpha_{n(k)}^2 \sum_{(n(k), n(k+1)]} |(\varphi_i, \varphi_j)| \leqq \sum_{(n(k), n(k+1)]} |(\varphi_i, \varphi_j)| \alpha_i \alpha_j;$$

hence

$$\sum_{k=1}^{N} d_{n(k)} \leqq \sum_{(n(1), n(N)]} |(\varphi_i, \varphi_j)| \alpha_i, \alpha_j + \alpha_{n(N)}^2 \sum_{i,j \geqq n(N)+1} |(\varphi_i, \varphi_j)| \leqq$$

$$\leqq \sum_{i,j \geqq n(1)} |(\varphi_i, \varphi_j)| \alpha_i \alpha_j < +\infty.$$

Hence the series $\Sigma d_{n(k)}$ converges, and this implies by Beppo Levi's theorem that $s_{n(k)} \to f$ a.e. as $k \to \infty$.

Theorem 3. *If $\{\varphi_i\}_{i=1}^{\infty} \subset L_2$ and*

$$\sum_{i,j=1}^{\infty} |(\varphi_i, \varphi_j)| \log i \log j < +\infty$$

then $s_n(x) = \sum_{i=1}^{n} \varphi_i(x)$ converges a.e. (cf. [8]).

Proof. Theorem 2 with $\alpha_i = \log i$ yields the convergence of the sequence $s_{2^m}(x)$ as $m \to \infty$. It is sufficient to show that

$$\max_{2^m < n \leqq 2^{m+1}} |s_n(x) - S_{2^m}(x)| = \sigma_n(1) \quad \text{a.e. as} \quad n \to \infty,$$

Applying Corollary of Theorem 1 with $\varphi_{2^m+1}, \varphi_{2^m+2}, \ldots, \varphi_{2^{m+1}}$, we have

$$|s_n(x) - s_{2^m}(x)| \leqq \delta_{2^m}(x) \quad \text{for} \quad 2^m < n \leqq 2^{m+1}.$$

$$M(\delta_{2^m}^2) = o\left((\log 2^m)^2\right) \sum_{i,j=2^m+1}^{2^{m+1}} |(\varphi_i, \varphi_j)| =$$

$$= o(1) \sum_{i,j=2^m+1}^{2^{m+1}} |(\varphi_i, \varphi_j)| \log i \log j,$$

whence our assertion follows by Beppo Levi's theorem.

Corollary. *If* $\{\varphi_i\}_{i=1}^{\infty} \subset L_2$, $\varrho(k)$ *is a non-negative real valued function satisfying*

$$\sum_{k=1}^{\infty} k\varrho(k) < +\infty \quad \textit{and} \quad |(\varphi_i, \varphi_j)| \leqq \frac{\varrho(i+j)}{\log i \log j},$$

then the series $\sum_{i=1}^{\infty} \varphi_i$ *converges a.e.*

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