Remark to a paper of Gaposhkin

By J. KOMLÓS and P. RÉVÉSZ in Budapest

In [1] GAPOSHKIN proved the following:

Theorem A. *Let* $\{\varphi_i\}$ *be a sequence of measurable functions defined on a measure space* $\{X, S, \mu\}$ *of finite measure for which*

$$\int \varphi_i^4 \leqq K, \tag{1}$$

$$\int \varphi_i \varphi_j \varphi_k \varphi_l = 0, \tag{2}$$

$$\int \varphi_i^2 \varphi_j \varphi_k = 0 \tag{3}$$

where i, j, k, l *are different integers. Then* $\{\varphi_i\}$ *is a convergence system, i.e. every series*

$$\sum_i c_i \varphi_i \quad \textit{statisfying} \quad \sum_i c_i^2 < \infty$$

is convergent almost everywhere in every arrangement of its terms.

The proof was based on the following result of STEČKIN:

Theorem B. (See [2], p. 27—31.) *Let* $\{\varphi_k\}$ *be a sequence of measurable functions defined on a measure space of finite measure for which*

$$\int \left(\sum_{k=1}^{N} c_k \varphi_k \right)^4 \leqq A \left(\sum_{k=1}^{N} c_k^2 \right)^2 \qquad (N = 1, 2, \ldots) \tag{4}$$

(for any sequence $\{c_k\}$ *of real numbers) where* A *is a positive constant. Then* $\{\varphi_k\}$ *is a convergence system.*

Making use of Theorem B in order to prove Theorem A it was enough to show that (1), (2) and (3) imply (4).

The aim of the present paper is to prove that condition (3) can be omitted, i.e. we prove the following:

Theorem. *Let* $\{\varphi_k\}$ *be a sequence of measurable functions defined on a measure space of finite measure, for which* (1) *and* (2) *hold. Then* $\{\varphi_k\}$ *is a convergence system.*

The proof of this theorem is also based on Theorem B, i.e. we prove that (1) and (2) (without (3)) imply (4). Notably we prove the following:

Lemma.[1]) *Let* $\{\varphi_k\}$ *be a sequence of measurable functions defined on a measure space of finite measure obeying conditions* (1) *and* (2). *Then*

$$\int\left(\sum_{i=1}^{N} c_i\varphi_i\right)^4 \leqq 17^2K\left(\sum_{i=1}^{N} c_i^2\right)^2 \qquad (N=1,2,\ldots)$$

for any sequence $\{c_i\}$ *of real numbers.*

Proof. Introduce the notations

$$A=A_n=\Big|\sum_{i,j,k} c_i^2c_jc_k\int\varphi_i^2\varphi_j\varphi_k\Big|, \qquad B=B_n=\Big|\sum_{i,j} c_i^3c_j\int\varphi_i^3\varphi_j\Big|,$$

$$M=\max(A,B), \quad \text{and} \quad \psi_i=c_i\varphi_i,$$

where i,j,k take all (not necessarily different) integers between 1 and n.

We have

$$(\sum_i\psi_i)^4=(\sum_{i,j}\psi_i\psi_j)^2=\sum_{i=1}^{n}\psi_i^4+6\sum_{1\leqq i<j\leqq n}\psi_i^2\psi_j^2+4\sum_{1\leqq i<j\leqq n}(\psi_i^3\psi_j+\psi_i\psi_j^3)+$$
$$+12\sum_{1\leqq i<j<k\leqq n}(\psi_i^2\psi_j\psi_k+\psi_i\psi_j^2\psi_k+\psi_i\psi_j\psi_k^2)+4!\sum_{1\leqq i<j<k<l\leqq n}\psi_i\psi_j\psi_k\psi_l,$$

$$\sum_{i,j,k}\psi_i^2\psi_j\psi_k=\sum_{i=1}^{n}\psi_i^4+2\sum_{1\leqq i<j\leqq n}(\psi_i^3\psi_j+\psi_i\psi_j^3)+2\sum_{1\leqq i<j\leqq n}\psi_i^2\psi_j^2+$$
$$+2\sum_{1\leqq i<j<k\leqq n}(\psi_i^2\psi_j\psi_k+\psi_i\psi_j^2\psi_k+\psi_i\psi_j\psi_k^2)$$

and

$$\sum_{i,j}\psi_i^3\psi_j=\sum_{i=1}^{n}\psi_i^4+\sum_{1\leqq i<j\leqq n}(\psi_i^3\psi_j+\psi_i\psi_j^3);$$

hence

$$\left(\sum_{i=1}^{n}\psi_i\right)^4=6\sum_{i,j,k}\psi_i^2\psi_j\psi_k-8\sum_{i,j}\psi_i^3\psi_j+3\sum_{i=1}^{n}\psi_i^4-\tag{5}$$
$$-6\sum_{1\leqq i<j\leqq n}\psi_i^2\psi_j^2+4!\sum_{1\leqq i<j<k<l\leqq n}\psi_i\psi_j\psi_k\psi_l.$$

[1]) This lemma was essentially formulated previously by R. Y. Serfling [3] but his proof is not quite complete.

Now we have

$$A = \Big|\sum_{i,j,k}\int \psi_i^2\psi_j\psi_k\Big| = \Big|\sum_{i=1}^{n}\int \psi_i^2 \sum_{j,k}\psi_j\psi_k\Big| \leqq$$

$$\leqq \sum_{i=1}^{n}\sqrt{\int\psi_i^4}\sqrt{\int\Big(\sum_{j,k}\psi_j\psi_k\Big)^2} \leqq \sqrt{K}\left(\sum_{i=1}^{n}c_i^2\right)\sqrt{\int\Big(\sum_{j,k}\psi_j\psi_k\Big)^2} \leqq$$

$$\leqq \sqrt{K}\left(\sum_{i=1}^{n}c_i^2\right)\sqrt{6A+8B+3K\left(\sum_{i=1}^{n}c_i^2\right)^2}$$

and

$$A^2 \leqq 8K\left(\sum_{i=1}^{n}c_i^2\right)^2\left(A+B+K\left(\sum_{i=1}^{n}c_i^2\right)^2\right). \tag{6}$$

Similarly

$$B = \Big|\int\sum_{i,j}\psi_i^3\psi_j\Big| = \Big|\sum_{i=1}^{n}\int\psi_i^2\sum_{j=1}^{n}\psi_i\psi_j\Big| \leqq \sum_{i=1}^{n}\sqrt{\int\psi_i^4}\sqrt{\int\left(\sum_{j=1}^{n}\psi_i\psi_j\right)^2} \leqq$$

$$\leqq \sqrt{K}\sum_{i=1}^{n}c_i^2\sqrt{\int\left(\sum_{j=1}^{n}\psi_i\psi_j\right)^2} \leqq \sqrt{K}\sqrt{\sum_{i=1}^{n}c_i^4}\sqrt{\sum_{i=1}^{n}\int\psi_i^2\left(\sum_{i=1}^{n}\psi_j\right)^2} =$$

$$= \sqrt{K}\sqrt{\sum_{i=1}^{n}c_i^4}\sqrt{\int\sum_{i,j,k}\psi_i^2\psi_j\psi_k} = \sqrt{K}\sqrt{\sum_{i=1}^{n}c_i^4}\sqrt{A}$$

and

$$B^2 \leqq 8K\left(\sum_{i=1}^{n}c_i^2\right)^2\left(A+B+K\left(\sum_{i=1}^{n}c_i^2\right)^2\right). \tag{7}$$

(6) and (7) together imply:

$$M^2 \leqq 8K\left(\sum_{i=1}^{n}c_i^2\right)^2\left(2M+K\left(\sum_{i=1}^{n}c_i^2\right)^2\right),$$

i.e.

$$\left(M-8K\left(\sum_{i=1}^{n}c_i^2\right)^2\right)^2 \leqq \left(\sum_{i=1}^{n}c_i^2\right)^4(64K^2+8K^2)$$

and

$$M \leqq \left(\sum_{i=1}^{n}c_i^2\right)^2 K(\sqrt{72}+8K) \leqq 17K\left(\sum_{i=1}^{n}c_i^2\right)^2.$$

(5) and (8) immediately prove our Lemma, and hence our Theorem too.

Let us mention that our condition (2) cannot be replaced by the condition $\int\psi_i\psi_j\psi_k=0$. More precisely, we make the following

Remark 1. *Let $\{c_k\}$ be a decreasing sequence of real numbers for which*

$$\sum_k c_k^2 \log^2 k = \infty,$$

then one can construct a uniformly bounded sequence $\{\varphi_k\}$ of measurable functions for which

$$\int \varphi_k^2 = 1, \quad \int \varphi_i = \int \varphi_i \varphi_j = \int \varphi_i \varphi_j \varphi_k = 0$$

(where the indices i, j, k are different), and the series

$$\sum c_k \varphi_k$$

is nowhere convergent.

The proof of this remark is based on the following theorem of TANDORI ([4]):

Theorem C. *If $c_1, c_2, \ldots$ is a decreasing sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2 \log^2 k = \infty,$$

then there exists a uniformly bounded sequence $\{\Phi_k\}$ of Lebesgue measurable functions on the interval $[0, 1]$ *such that*

$$\int \Phi_i = \int \Phi_i \Phi_j = 0, \quad \int \Phi_i^2 = 1,$$

and the series

$$\sum_{k=1}^{\infty} c_k \Phi_k$$

is nowhere convergent.

Now the proof of our remark runs by the following construction:

Let $\{\Phi_k\}$ be a sequence with the properties mentioned in Theorem C. Define the sequence $\{\varphi_k\}$ on the interval $[0, 2]$ by

$$\varphi_k(x) = \begin{cases} \dfrac{\Phi_k(x)}{\sqrt{2}} & \text{for} \quad x \in [0, 1), \\ \dfrac{-\Phi_k(x-1)}{\sqrt{2}} & \text{for} \quad x \in [1, 2]. \end{cases}$$

This sequence obviously satisfies all conditions of our remark.

Remark 2. A very easy evalution shows that our theorem remains correct if condition (2) is replaced by the following one: there exists a function $B(k)$ $(k=1, 2,\dots)$ for which

$$\left|\int \varphi_i \varphi_j \varphi_k \varphi_l\right| \leqq \min\big(B(l-k),\ B(j-i)\big) \quad (1 \leqq i<j<k<l)$$

and

$$\sum_{k=1}^{\infty} kB(k) < \infty.$$

This result is much stronger than that of Révész [5].

Added in proof: Prof. V. F. Gaposhkin informed us in a letter that he also proved the above theorem.

References

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