Direct product in locally finite categories

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A category will be called *locally finite* if only a finite number of morphisms joins any two objects of it. We are going to study "cancellation" properties of direct products in such categories. The results are generalizations of those of [1, 2] and no essentially new idea is used; however, the questions and results extend to categories in such a natural and general form that it seems to be worth stating them in a short note.[1])

Given a category $\mathcal{K}$, we denote by $H(A, B)$ the set of all morphisms of $\mathcal{K}$ from A to B.

Lemma 1. *Let A, B be objects of the locally finite category $\mathcal{K}$, and assume that there are monomorphisms φ from A to B and η from B to A. Then both φ and η are isomorphisms.*

Proof. Consider the morphisms $(\varphi\eta)^n$ $(n=1, 2, \ldots)$. Since $\mathcal{K}$ is locally finite there exist $k, m>0$ such that

$$(\varphi\eta)^k=(\varphi\eta)^{k+m}.$$

Now $\varphi\eta$ being a monomorphism, this implies

$$(\varphi\eta)^m=id_A, \tag{1}$$

i.e. putting $\eta'=\eta(\varphi\eta)^{m-1}$, we have $\varphi\eta'=\mathrm{id}_A$. Multiplication of (1) from the left by η gives $(\eta\varphi)^m\eta=\eta\cdot(\varphi\eta)^m=\eta$; since η is a monomorphism, it follows that

$$(\eta\varphi)^m=\eta'\varphi=id_B.$$

Hence η' is the inverse of φ and thus φ is an isomorphism. Similarly η is an isomorphism.

Remark. Obviously, Lemma 1 remains true if we consider only the subcategory determined by A and B.

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[1]) Recently A. Pultr (Prague) informed me that he also remarked the possibility of this generalization and obtained similar results.

Let $M(A, B)$ be the set of all monomorphisms of A to B. If P is an equivalence relation on the set $S(X)$ of all morphisms to X, we denote by $H(P, A)$ the set of those morphisms $\varphi \in H(X, A)$ which satisfy $\alpha\varphi = \alpha'\varphi$ for every $(\alpha, \alpha') \in P$.

Lemma 2. *Let A, B, X be objects in the locally finite category $\mathcal{K}$ and assume $|H(P, A)| = |H(P, B)|$ for every equivalence-relation P on $S(X)$. Then $|M(X, A)| = |M(X, B)|$.*

Proof. We may assume $\mathcal{K}$ and thus $S(X)$ are finite. Obviously, $\varphi \in M(X, A)$ iff $\varphi \notin H(P, A)$ except P is the identity relation j on $S(X)$. Hence, by sieving we get

$$(2) \qquad |M(X, A)| = \sum_{k \geqq 0} (-1)^k \sum_{P_1, \dots, P_k \neq j} |H(P_1, A) \cap \dots \cap H(P_k, A)| =$$

$$= \sum_{k \geqq 0} (-1)^k \sum_{P_1, \dots, P_k \neq j} |H(P_1 \vee \dots \vee P_k, A)|,$$

where $P_1 \vee \dots \vee P_k$ means the least equivalence-relation containing $P_1 \cup \dots \cup P_k$ (the member corresponding to $k=0$ is $|H(j, A)| = |H(X, A)|$).

Now $|M(X, B)|$ can also be expressed by a formula like (2) and the two right hand sides are equal by assumption. Hence the statement follows.

Remark. If the condition of the lemma holds for $X=A$ and $X=B$ then $|M(A, B)| = |M(A, A)| > 0$ and $|M(B, A)| = |M(B, B)| > 0$, thus by Lemma 1, A and B are isomorphic.

Lemma 3. *Let (π_1, π_2) be a (projective) direct product; $\pi_1 \in H(AB, A)$, $\pi_2 \in H(AB, B)$. Then for any object X and equivalence-relation P on $S(X)$, $|H(P, AB)| = |H(P, A)| \cdot |H(P, B)|$.*

Proof. It is easy to verify that a $\varphi \in H(X, AB)$ belongs to $H(P, AB)$ iff $\varphi\pi_1 \in H(P, A)$ and $\varphi\pi_2 \in H(P, B)$. Hence the proposition follows.

We prove now that the kth root is unique in any locally finite category.

Theorem 1. *Let $(\pi_1, \dots, \pi_k)$ and $(\varrho_1, \dots, \varrho_k)$ be two (projective) direct decompositions of the same object C of the locally finite category $\mathcal{K}$ and let $\pi_1, \dots, \pi_k \in H(C, A)$, $\varrho_1, \dots, \varrho_k \in H(C, B)$ (Fig. 1). Then A and B are isomorphic.*

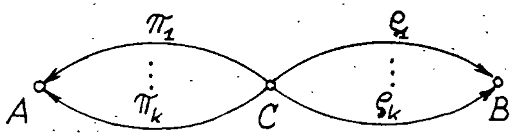

Fig. 1

Proof. By Lemma 3 we have $|H(P, C)|=|H(P, A)|^k=|H(P, B)|^k$ for any equivalence relation P on any $S(X)$. Hence $|H(P, A)|=|H(P, B)|$. By the remark following Lemma 2, this implies that A, B are isomorphic.

A question analogous to Theorem 1 is whether the following diagram (Fig. 2) implies that A and B are isomorphic (ι is isomorphism, $\pi_1, \pi_2, \varrho_1, \varrho_2$ direct products).

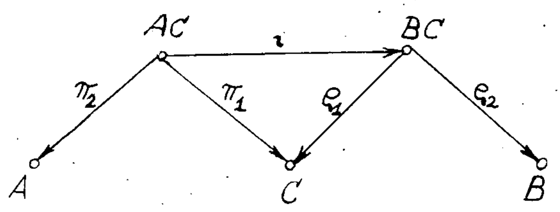

Fig. 2

This is not the case in general but we have

Theorem 2. *If in Fig. 2 both A and B have morphisms into C then they are isomorphic.*

Proof. Let P be an equivalence relation on $S(X)$ where X is some object. By Lemma 3,

$$|H(P, A)|\cdot|H(P, C)| = |H(P, AC)| = |H(P, BC)| = |H(P, B)|\cdot|H(P, C)|.$$

Now if $H(P, C)\neq\emptyset$ then $|H(P, A)|=|H(P, B)|$. But this also follows if $|H(P, C)|=\emptyset$, since then both $H(P, A)$ and $H(P, B)$ are empty. Hence by Lemma 2, A and B are isomorphic.

Now we consider the case when Theorem 2 cannot be applied.

Theorem 3. *In the diagram of Fig. 3, AD and BD are isomorphic* ((π_1, π_2), (ϱ_1, ϱ_2), (σ_1, σ_2), (τ_1, τ_2) *are direct products, ι is an isomorphism*).

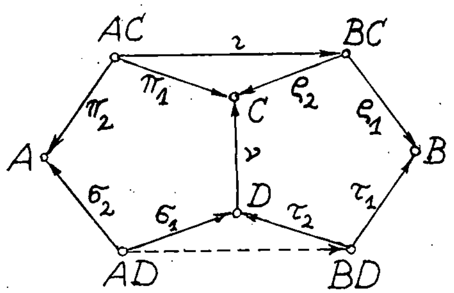

Fig. 3

The proof is very similar to that of Theorem 2, therefore we omit it.

Theorem 4. *In the diagram of Fig. 2 one can always find an isomorphism ι such that the diagram commutes.*

Proof. Denote by $\mathscr{K}^*$ the category of those morphisms φ which are in $H(AC, AC)$ and satisfy $\varphi\pi_1=\pi_1$; or in $H(AC, BC)$ and satisfy $\varphi\varrho_1=\pi_1$; or in $H(BC, AC)$ and satisfy $\varphi\pi_1=\varrho_1$; or in $H(BC, BC)$ and satisfy $\varphi\varrho_1=\varrho_1$. It is easily seen that these morphisms form a category indeed. We denote the set of morphisms, monomorphisms etc. in $\mathscr{K}^*$ by $H^*(X, Y)$, $M^*(X, Y)$ etc.

Let P be an equivalence relation on $S(X)$ where $X=AC$ or BC. It is enough to show

$$|H^*(P, AC)=|H^*(P, BC)|, \tag{3}$$

since then by the remark after Lemma 2 the statement follows.

Consider the pairs (δ, φ) where $\delta\in H(X, C)$ and $\varphi\in H^*(P, AB)$. Their number is $|H(X, C)|\cdot|H^*(P, AC)|$. We attach to every such (δ, φ) an (ε, ψ) where $\varepsilon\in H(X, C)$ and $\psi\in H^*(P, BC)$. Let φ^* be defined by

$$\varphi^*\pi_1=\delta, \quad \varphi^*\pi_2=\varphi\pi_2,$$

and set

$$\psi^*=\varphi^*\iota, \quad \varepsilon=\psi^*\varrho_1.$$

Define ψ by

$$\psi\varrho_1=\begin{cases}\pi_1 & \text{if } X=AC,\\ \varrho_1 & \text{if } X=BC,\end{cases} \qquad \psi\varrho_2=\psi^*\varrho_2.$$

Then, obviously, $\psi\in H^*(P, BC)$ and $\varepsilon\in H(X, C)$, and the correspondence is one-to-one since the argument defining (ε, ψ) can be converted. Hence the number of pairs (ε, ψ) $(\varepsilon\in H(X, C)$, $\psi\in H^*(P, BC)$ is also $|H(X, C)|\cdot|H^*(P, AC)|$, and since $H(X, C)\neq\emptyset$, (3) follows.

References

[1] L. Lovász, Operations with structures, *Acta Math. Acad. Sci. Hung.*, **18** (1967), 321—328.
[2] L. Lovász, On the cancellation law among finite relational structures, *Periodica Math. Hung.*, **1** (1971), 145—156.

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