On convergence properties of operators of class $\mathscr{C}_\varrho$

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In the preceding paper[1]) G. ECKSTEIN proved that if a (bounded, linear) operator T on a (complex) Hilbert space $\mathfrak{H}$ belongs to a class $\mathscr{C}_\varrho$ ($\varrho>0$)[2]) then $\|T^{*n}f\|$ converges as $n\to\infty$ for every $f\in\mathfrak{H}$. We are going to give another proof of the same statement, and of some related convergence properties.

We assume once for all that T is of a class $\mathscr{C}_\varrho$ ($\varrho>0$) so that it has a unitary ϱ-dilation on some Hilbert space $\mathfrak{K}(\supset\mathfrak{H})$, i.e. a unitary operator U such that

$$T^n f=\varrho PU^n f \quad \text{for} \quad f\in\mathfrak{H} \quad \text{and} \quad n=1,2,\ldots,$$

P being the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. We set $\mathfrak{M}_+=\bigvee_{n=0}^{\infty} U^n\mathfrak{H}$.

The following lemma is crucial for our purposes:

Lemma. *If $h\in U^{n+1}\mathfrak{M}_+$ for some $n\geqq 0$ then $Ph=T^nPU^{-n}h$.*

Proof. Since $U^{n+1}\mathfrak{M}_+$ is spanned by the elements of the form $h=U^{n+i}f$ ($f\in\mathfrak{H}$; $i\geqq 1$) it suffices to consider such an h. Then,

$$Ph=PU^{n+i}f=\frac{1}{\varrho}T^{n+i}f=\frac{1}{\varrho}T^nT^if=\frac{1}{\varrho}T^n\cdot\varrho PU^if=$$

$$=\frac{1}{\varrho}T^n\cdot\varrho PU^{-n}U^{n+i}f=T^nPU^{-n}h,$$

and the proof is done.

Denote by Q_n the projection onto $U^{n+1}\mathfrak{M}_+$. Then $Q=\lim Q_n$ exists and is the projection onto $\bigcap_{n=1}^{\infty} U^n\mathfrak{M}_+$. It follows from the lemma that $PQ_nh=T^nPU^{-n}Q_nh$ for every $h\in\mathfrak{K}$. Consequently, if $f\in\mathfrak{H}$, then $(h,Q_nf)=(PQ_nh,f)=(T^nPU^{-n}Q_nh,f)=$ $=(h,Q_nU^nT^{*n}f)$. It results that

$$Q_nf=Q_nU^nT^{*n}f \quad \text{for} \quad f\in\mathfrak{H} \quad \text{and} \quad n=1,2,\ldots.$$

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[1]) Sur les opérateurs de classe $\mathscr{C}_\varrho$, *Acta Sci. Math.*, **33** (1972), 345—352.

[2]) For references on $\mathscr{C}_\varrho$ classes see: B. SZ.-NAGY and C. FOIAŞ, *Harmonic Analysis of Operators on Hilbert Space* (London—Amsterdam—Budapest, 1970).

Now since $U^n T^{*n} f \in U^n \mathfrak{M}_+$ $(n \geqq 0)$ we have for $g \in \mathfrak{K}$:

$$(U^n T^{*n} f, g) = (Q_n f, g) + (U^n T^{*n} f, (Q_{n-1} - Q_n) g).$$

The sequence $T^{*n} f$ is bounded and $Q_{n-1} - Q_n \to 0$. Hence,

(1) $$U^n T^{*n} f \to Qf \quad as \quad n \to \infty, \quad weakly\ for\ every \quad f \in \mathfrak{H}.$$

As $(U^n T^{*n} f, f) = \varrho\ (T^n T^{*n} f, f)$ for $n \geqq 1$, we deduce from (1) that $\|T^{*n} f\|^2 \to \to \varrho \|Qf\|^2$. Thus we have proved:

(2) $$\|T^{*n} f\|^2 \to \varrho \|Qf\|^2 \quad as \quad n \to \infty, \quad for\ every \quad f \in \mathfrak{H}.$$

As T^* is of class $\mathscr{C}_\varrho$ whenever T is, we have got a sharpening of ECKSTEIN's result.

As weak convergence $u_n \to u$ implies strong convergence if and only if $\|u_n\| \to \|u\|$, we infer from (1) and (2) that the convergence (1) holds for some f in the strong sense too if and only if $\|Qf\|^2 = \varrho \|Qf\|^2$. This is the case for every f if $\varrho = 1$, and for f satisfying $Qf = 0$ if $\varrho \neq 1$.

It is easy to give an example of operator T in $\mathscr{C}_\varrho$ $(\varrho > 1)$ for which $Qf = 0$ if $f = 0$. This is indeed the case for every unitary T since then $\|f\|^2 = \lim \|T^{*n} f\|^2 = = \varrho \|Qf\|^2$. Thus, in general, (1) does not hold true for strong convergence.

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