On the convergence of function series

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Dedicated to Béla Szőkefalvi-Nagy on his 60th birthday

1. Let X be a measurable space with a positive measure μ and $\{f_n(x)\}$ a sequence of μ-measurable functions in X. On the measurable set $E \subset X$, consider the Lebesgue functions of the system $\{f_n(x)\}$:

$$L_n(x) = \int_E \left| \sum_{k=0}^{n} f_k(x) f_k(y) \right| d\mu(y),$$

and for an index sequence $\nu_1 < \nu_2 < \dots$ set

$$L_{\nu_n}(E) = \int_E \max_{0 \leqq j \leqq n} L_{\nu_j}(x)\, d\mu(x).$$

Recently we have proved the following theorem ([2], Theorem 2):

If E is of finite measure and $L_n(E) \leqq K$ $(n=0, 1, \dots)$, further if $\{a_n\}$ is a sequence of real numbers such that $\sum a_n^2 < \infty$, then the series $\sum a_n f_n(x)$ converges on E a.e.

If no more than the uniform boundedness of the subsequence $\{L_{\nu_n}(E)\}$ is required then for the subsequence $\{s_{\nu_n}(x)\}$ of the partial sums

$$s_{\nu_n}(x) = \sum_{k=0}^{\nu_n} a_k f_k(x)$$

a similar statement could be proved only under a rather restrictive subcondition ([2], Theorem 3). But it seems that an analogous statement without any restriction could have a certain importance. In the following we shall prove it by suppressing, besides the mentioned subcondition, also the inutile condition that E should have a finite measure. More exactly, we shall prove the following

Theorem 1. *Let $\{a_n\}$ be an arbitrary sequence of real numbers with $\sum a_n^2 < \infty$ and $\{f_n(x)\}$ an arbitrary sequence of μ-integrable functions defined on the measurable set $E \subset X$. Then the condition $L_{\nu_n}(E) \leqq K$ $(n=1, 2, \dots)$ implies the convergence of the sequence $\{s_{\nu_n}(x)\}$ on E a.e.*

Theorem 1 has different consequences of various kind; one of them could open a new way to the study of the convergence properties of certain function series even if the corresponding Lebesgue functions do not form a bounded sequence. One of these consequences concerns the series of weakly multiplicative functions, a notion we introduced occasionally [3] and which is a vigorous generalization of the stochastically independent functions.

Definition. A system $\{\varphi_n(x)\}$ of μ-integrable functions on E is called *weakly multiplicative*, if the integrals $\int_E \varphi_{\nu_1}(x)\varphi_{\nu_2}(x)\dots\varphi_{\nu_n}(x)\,d\mu(x)$ exist for all finite collections of indices $\nu_1<\nu_2<\dots<\nu_n$ and

$$\sum\left|\int_E \varphi_{\nu_1}(x)\,\varphi_{\nu_2}(x)\dots\,\varphi_{\nu_n}(x)\,d\mu(x)\right|<\infty,$$

where the summation has to be taken for all finite collections of $\nu_1<\nu_2<\dots<\nu_n$.

We shall prove the convergence a.e. of the series $\sum c_n\varphi_n(x)$ if $\sum c_n^2<\infty$ and $\{\varphi_n(x)\}$ is bounded. (We have already proved this [3] assuming the validity of our present Theorem 1.) Then we shall study also the absolute convergence of such series.

The convergence a.e. of $\sum c_n\varphi_n(x)$ under $\sum c_n^2<\infty$ generalizes a theorem we proved earlier ([1], Theorem 1). Our present result is much stronger than that earlier one; this can be seen by the following fact: FIEDLER and TRAUTNER proved [4] the existence of a complete bounded orthonormal system which does not contain any infinite subsystem of multiplicatively orthogonal functions (to which our earlier theorem refers). Moreover, FRIESS and TRAUTNER [5] proved that the bounded complete orthogonal systems containing an infinite multiplicatively orthogonal subsystem are in some sense "rare". Whereas we shall see that every bounded infinite orthonormal system contains an infinite weakly multiplicative subsystem.

2. Turning to the proof of our Theorem 1 we first prove an inequality which plays a similar role as the Rademacher—Menchov inequality in the theory of orthogonal series.

Let $n\geqq m$ be two fixed positive integers and denote by $m(x)$ and $n(x)$ measurable functions taking only integer values between m and n, i.e. $m\leqq m(x)\leqq n(x)\leqq n$. If $r_n(t)$ denotes the kth Rademacher function defined in $0\leqq t\leqq 1$, i.e. $r_k(t)=$ $=\operatorname{sign}\sin 2^n\pi t$, we can write

$$s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)=\int_0^1\sum_{k=\nu_m}^{\nu_n}a_k r_k(t)\sum_{k=\nu_{m(x)}+1}^{\nu_{n(x)}}r_k(t)f_k(x)\,dt.$$

Hence, denoting by P and N the sets on which $s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)\geqq 0$ or <0, respectively, we get by Schwarz's inequality

$$I_{m,n}(P)=\int_P [s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)]\,d\mu(x)\leqq$$

$$\leqq\left\{\int_0^1\left[\sum_{k=\nu_m}^{\nu_n} a_k r_k(t)\right]^2 dt\int_0^1\left[\int_P \sum_{k=\nu_{m(x)}+1}^{\nu_{n(x)}} r_k(t)f_k(x)\,d\mu(x)\right]^2 dt\right\}^{\frac{1}{2}}=$$

$$=\left\{\sum_{k=\nu_m}^{\nu_n} a_k^2\int_P\int_P\int_0^1 \sum_{k=\nu_{m(x)}+1}^{\nu_{n(x)}} r_k(t)f_k(x)\sum_{k=\nu_{m(y)}+1}^{\nu_{n(y)}} r_k(t)f_k(y)\,dt\,d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}}=$$

$$=\left\{\sum_{k=\nu_m}^{\nu_n} a_k^2\int_P\int_P \sum_{k=\nu_{m(x,y)}+1}^{\nu_{n(x,y)}} f_k(x)f_k(y)\,d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}},$$

where $\nu_{m(x,y)}=\max\{\nu_{m(x)},\nu_{m(y)}\}$ and $\nu_{n(x,y)}=\min\{\nu_{n(x)},\nu_{n(y)}\}$. Write the sum in the last integral in the following form:

$$\sum_{k=\nu_{m(x,y)}+1}^{\nu_{n(x,y)}}=\sum_{k=0}^{\nu_{n(x,y)}}-\sum_{k=0}^{\nu_{m(x,y)}}.$$

Then we get by definition of $L_{\nu_n}(x)$ and $L_{\nu_n}(E)$

$$I_{m,n}(P)\leqq\left\{\sum_{k=\nu_m}^{\nu_n} a_k^2\int_P\int_P\left(\left|\sum_{k=0}^{\nu_{n(x,y)}} f_k(x)f_k(y)\right|+\left|\sum_{k=0}^{\nu_{m(x,y)}} f_k(x)f_k(y)\right|\right)d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}}\leqq$$

$$\leqq\left\{\sum_{k=\nu_m}^{\nu_n} a_k^2\left(2\int_P\int_P\left|\sum_{k=0}^{\nu_{n(x)}} f_k(x)f_k(y)\right|d\mu(x)\,d\mu(y)+\right.\right.$$

$$\left.\left.+2\int_P\int_P\left|\sum_{k=0}^{\nu_{m(x)}} f_k(x)f_k(y)\right|d\mu(x)\,d\mu(y)\right)\right\}^{\frac{1}{2}}\leqq$$

$$\leqq\left\{\sum_{k=\nu_m}^{\nu_n} a_k^2\left(2\int_P L_{\nu_{n(x)}}(x)\,d\mu(x)+2\int_P L_{\nu_{m(x)}}(x)\,d\mu(x)\right)\right\}^{\frac{1}{2}}\leqq\left\{4\sum_{k=\nu_m}^{\nu_n} a_k^2\cdot L_{\nu_n}(E)\right\}^{\frac{1}{2}}.$$

The same estimate holds true for the integral of $s_{\nu_{m(x)}}(x)-s_{\nu_{n(x)}}(x)$ extended over the set N, so we get finally

$$\int_E |s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)|\,d\mu(x)\leqq\left\{16L_{\nu_n}(E)\sum_{k=\nu_m}^{\nu_n} a_k^2\right\}^{\frac{1}{2}}. \tag{1}$$

This is the inequality we intended to prove.

3. From (1) the proof of Theorem 1 follows. Indeed, choose for $m(x)$ the least integer $\geqq m$ and for $n(x)$ the largest integer $\leqq n$ such that

$$|s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)|=\max_{m\leqq i\leqq j\leqq n}|s_{\nu_j}(x)-s_{\nu_i}(x)|.$$

Denote by $A_{m,n}$ the set on which

$$|s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)|\geqq\varepsilon \qquad (\varepsilon>0).$$

From (1) and the inequality

$$\varepsilon|A_{m,n}|\leqq\int_E|s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)|\,d\mu(x)$$

one gets the estimate

$$|A_{m,n}|\leqq\varepsilon^{-1}\left\{16L_{\nu_n}(E)\sum_{k=\nu_m}^{\nu_n}a_k^2\right\}^{\frac{1}{2}},$$

where $|A_{m,n}|$ denotes the μ-measure of $A_{m,n}$. Since $\sum a_k^2<\infty$, for every $\varepsilon>0$ there exists an index m_ε such that

(2) $$\sum_{k=\nu_m}^{\nu_n}a_k^2<\frac{\varepsilon^4}{16K} \qquad (m\geqq m_\varepsilon),$$

where K is the common bound of the numbers $L_{\nu_n}(E)$. Hence $|A_{m,n}|<\varepsilon$ for every $m\geqq m_\varepsilon$ and $n\geqq m$. From the definition of $m(x)$ and $n(x)$ it follows that, for m fixed, the sequence $\{|s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)|\}$ is not decreasing if $n\to\infty$. Then, for m fixed, the sequence of sets $\{A_{m,n}\}$ is also not decreasing. Therefore the set

$$A(m)=\lim_{n\to\infty}A_{m,n}$$

exists and has measure $|A(m)|\leqq\varepsilon$ for an arbitrary $m\geqq m_\varepsilon$.

Put $m_1>m$, then

$$|s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)|\geqq|s_{\nu_{n(x)}}(x)-s_{\nu_{m_1(x)}}(x)|,$$

hence $A(m_1)\subset A(m)$. Or if $x\notin A(m)$ we have

$$|s_{\nu_n}(x)-s_{\nu_m}(x)|\leqq|s_{\nu_{n(x)}}(x)-s_{\nu_{m(x)}}(x)|<\varepsilon$$

for an arbitrary $n\geqq m$. So we got finally the estimate

(3) $$|s_{\nu_n}(x)-s_{\nu_m}(x)|<\varepsilon$$

for every $m\geqq m_\varepsilon$ and an arbitrary $n\geqq m$, provided $x\notin A(m_\varepsilon)$. The measure of $A(m_\varepsilon)$ being $\leqq\varepsilon$, the inequality (3) holds true except the points of a set of measure $\leqq\varepsilon$.

Repeating the same order of ideas with $\varepsilon/2$, $\varepsilon/4$, ... instead of ε, we obtain a sequence of sets $A(m_{\varepsilon/2})$, $A(m_{\varepsilon/4})$, ... with measures $\leqq\varepsilon/2$, $\leqq\varepsilon/4$, ... on the complements of which (3) holds true with $\varepsilon/2$, $\varepsilon/4$, ... instead of ε. Form the set

$$A=\bigcup_{k=0}^{\infty}A(m_{\varepsilon/2^k}),$$

then $|A| \leqq 2\varepsilon$ and, for $x \notin A$, we have

$$|s_{\nu_n}(x) - s_{\nu_m}(x)| < \frac{\varepsilon}{2^k} \qquad (k = 0, 1, \ldots)$$

for every $n \geqq m \geqq m_{\varepsilon/2^k}$. This means that $\{s_{\nu_n}(x)\}$ converges except perhaps on the set A of measure $\leqq 2\varepsilon$ and the proof is complete.

4. We say that the function system $\{\varphi_n(x)\}$ can be extended to a $L_{\nu_n}(E)$-bounded system $\{f_n(x)\}$, if $f_{\nu_n+1}(x) = \varphi_n(x)$ and the system $\{f_n(x)\}$ has the property $L_{\nu_n}(E) \leqq K$ $(n = 1, 2, \ldots)$. From Theorem 1 we deduce immediately the following

Corollary. *If a system $\{\varphi_n(x)\}$ can be extended to a $L_{\nu_n}(E)$-bounded system $\{f_n(x)\}$, then the series $\sum c_n \varphi_n(x)$ converges on E a.e. under the sole condition $\sum c_n^2 < \infty$.*

Indeed, if we set $a_k = c_n$ for $k = \nu_n + 1$ $(n = 1, 2, \ldots)$ and $a_k = 0$ for every other k, then we have

$$\sum_{k=0}^{\nu_n+1} a_k f_k(x) = \sum_{k=0}^{n} c_k \varphi_k(x)$$

and the corollary follows from Theorem 1.

We would like to emphasize that this corollary contains eventually a possible way for the study of the convergence properties of different series $\sum c_n \varphi_n(x)$. Considering namely the circumstance that we do not need more than the μ-integrability of the functions $f_k(x)$, it might be possible that, by a suitable choice of the indices ν_n and the functions $f_k(x)$ which we insert between $\varphi_n(x)$ and $\varphi_{n+1}(x)$, one could extend different systems $\{\varphi_n(x)\}$ to a $L_{\nu_n}(E)$-bounded system $\{f_n(x)\}$, and so conclude the convergence a.e. of $\sum c_n \varphi_n(x)$ if $\sum c_n^2 < \infty$. It would be very interesting if one could apply this method to some classical orthogonal system.

5. We defined in Sec. 1 the notion of a weekly multiplicative system $\{\varphi_n(x)\}$. For such systems we can apply the above sketched method to prove the following

Theorem 2. *If $\{\varphi_n(x)\}$ is weakly multiplicative on the set $E \subset X$ of finite measure, further if $|\varphi_n(x)| \leqq M_n$ with $M_n \geqq 1$, then the condition $\sum c_n^2 M_n^2 < \infty$ implies the convergence of the series $\sum c_n \varphi_n(x)$ on E a.e.*

Denote by $\{\psi_n(x)\}$ the product system of $\{\varphi_n(x)/M_n\}$, i.e. $\psi_0(x) \equiv 1$ and $\psi_n(x) = (\varphi_{\nu_1+1}(x) \ldots \varphi_{\nu_k+1}(x))/(M_{\nu_1+1} \ldots M_{\nu_k+1})$ for $n = 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}$. Then

$$\psi_{2^{n-1}}(x) = \varphi_n(x)/M_n,$$

and it is easy to see that

$$\sum_{k=0}^{2^n-1} \psi_k(x)\psi_k(y) = \prod_{k=1}^{n} \left(1 + \frac{\varphi_k(x)\varphi_k(y)}{M_k^2}\right). \tag{4}$$

We want to show that the product system $\{\psi_n(x)\}$ is $L_{2^n-1}(E)$-bounded, hence $\{\varphi_n(x)/M_n\}$ is imbedded in a $L_{2^n-1}(E)$-bounded system. Taking into account $|\varphi_n(x)|/M_n \leqq 1$, the right hand side of (4) is non-negative; so we can omit the sign of absolute value in the integral defining $L_{2^n-1}(x)$, hence

$$L_{2^n-1}(x) = \int_E \sum_{k=0}^{2^n-1} \psi_k(x)\psi_k(y)\,d\mu(y) \leqq$$
$$\leqq \sum_{k=0}^{2^n-1} |\psi_k(x)| \left|\int_E \psi_k(y)\,d\mu(y)\right| \leqq \sum_{k=0}^{\infty} \left|\int_E \psi_k(y)\,d\mu(y)\right| \leqq C_1,$$

where $C_1, C_2, \ldots$ are absolute constants. The last inequality is a consequence of the weak multiplicativity of $\{\varphi_n(x)\}$. In fact, denoting by $\{\psi_n^*(x)\}$ the product system of $\{\varphi_n(x)\}$ we have by assumption

$$\sum_{n=0}^{\infty} \left|\int_E \psi_n^*(y)\,d\mu(y)\right| \leqq C_2$$

and, because of $M_n \geqq 1$,

$$\sum_{n=0}^{\infty} \left|\int_E \psi_n(y)\,d\mu(y)\right| = \sum_{n=0}^{\infty} \frac{1}{M_{\nu_1} M_{\nu_2} \ldots M_{\nu_n}} \left|\int_E \psi_n^*(y)\,d\mu(y)\right| \leqq$$
$$\leqq \sum_{n=0}^{\infty} \left|\int_E \psi_n^*(y)\,d\mu(y)\right| = C_2.$$

The sequence $\{L_{2^n-1}(x)\}$ being uniformly bounded on E, we get $L_{2^n-1}(E) \leqq C_3$ $(n=1, 2, \ldots)$ by the finiteness of $|E|$. Therefore we can apply our corollary to the series $\sum c_n M_n \dfrac{\varphi_n(x)}{M_n}$ and our statement follows.

The property to be a weakly multiplicative system is, of course, independent of the order of the terms. Hence, in the statement of Theorem 2 we can say *unconditional* convergence a.e. instead of simple convergence a.e. Theorem 2 immediately implies different forms of the strong law of great numbers (see [3]). But P. Révész [6] proved that also the law of iterated logarithm can be extended, in a proper form, to weakly independent systems.

6. Now we are looking for the absolute convergence of expansions in the functions $\varphi_n(x)$ of a weakly multiplicative system.

Theorem 3. *Let $\{\varphi_n(x)\}$ be a bounded weakly multiplicative system on the set $E \subset X$ of finite measure and assume*

$$\varliminf_{n\to\infty} \int_E |\varphi_n(x)|\,d\mu(x) \geqq q > 0. \tag{5}$$

If the μ-integrable function $f(x)$ is one-sided bounded and the expansion coefficients of $f(x)$ in the product functions $\psi_n(x)$

$$a_n = \int_E f(x)\psi_n(x)\,d\mu(x)$$

vanish except perhaps the coefficients

$$a_{2^{n-1}} = c_n = \int_E f(x)\varphi_n(x)\,d\mu(x),$$

then the series $\sum |c_n|$ is convergent.

We may assume without restricting the generality that $|\varphi_n(x)| \leqq 1$ for all n. Indeed we have $|\varphi_n(x)| \leqq C_4$ by assumption and the absolute convergence of the series $\sum c_n\varphi_n(x)$ is equivalent to that of $C_4 \sum c_n\varphi_n(x)/C_4$.

Rearrange $\{\varphi_n(x)\}$ in an arbitrary way: $\{\varphi_{\nu_k}(x)\}$, and put

$$s_n(\{\nu_k\}, x) = \sum_{k=1}^{n} c_{\nu_k}\varphi_{\nu_k}(x).$$

Denote by $\{\psi_n^*(x)\}$ the rearranged product system of $\{\varphi_n(x)\}$ corresponding to the arrangement $\{\varphi_{\nu_k}(x)\}$. Since the expansion coefficients of $f(x)$ in $\psi_k^*(x)$ vanish for $\psi_k^*(x) \neq \varphi_n(x)$, i.e. for $k \neq 2^{n-1}$, we get

$$(6)\qquad s_n(\{\nu_k\}, x) = \int_E f(t) \sum_{k=1}^{n} \varphi_{\nu_k}(t)\varphi_{\nu_k}(x)\,d\mu(t) = \int_E f(t) \sum_{k=0}^{2^n-1} \psi_k^*(t)\psi_k^*(x)\,d\mu(t).$$

By assumption $f(t)$ is bounded from one side, for instance $f(t) \leqq M$, so we infer from (6) and

$$\sum_{k=0}^{2^n-1} \psi_k^*(t)\psi_k^*(x) = \prod_{k=1}^{n} [1+\varphi_{\nu_k}(t)\varphi_{\nu_k}(x)] \geqq 0$$

the estimate

$$(7)\qquad s_n(\{\nu_k\}, x) \leqq M \sum_{k=0}^{2^n-1} |\psi_k^*(x)| \left|\int_E \psi_k^*(t)\,d\mu(t)\right| \leqq M \sum_{k=0}^{\infty} \left|\int_E \psi_k^*(t)\,d\mu(t)\right| \leqq C_5 M.$$

Furthermore, in a similar way we obtain

$$(8)\qquad -s_n(\{\nu_k\}, x) = \int_E f(t) \prod_{k=0}^{n} [1-\varphi_{\nu_k}(t)\varphi_{\nu_k}(x)]\,d\mu(t) \leqq$$
$$\leqq M \sum_{k=0}^{\infty} |\psi_k^*(x)| \left|\int_E \psi_k^*(t)\,d\mu(t)\right| \leqq C_5 M.$$

The estimates (7) and (8) give the result

$$|s_n(\{\nu_k\}, x)| \leqq C_5 M \qquad (n = 1, 2, \dots)$$

and this common bound holds good for every rearrangement of the series $\sum c_n\varphi_n(x)$.

Hence, according to a classical theorem of Riemann, the convergence of the series $\sum|c_n\varphi_n(x)|$ follows. So the sum of this series is bounded on any $E'\subseteqq E$ with $|E'|>0$ therefore

$$\sum_{n=1}^{\infty}|c_n|\int_{E'}|\varphi_n(x)|\,d\mu(x)<\infty,$$

and so we get by (5)

$$\frac{1}{q}\sum_{n=1}^{\infty}|c_n|<\infty$$

as we have stated.

7. In section 1 we mentioned that the bounded complete orthonormal systems containing an infinite multiplicatively orthogonal subsystem are "rare" in some sense. Now we will show that *every bounded infinite orthonormal system on a set of finite measure, even if it is not complete, contains an infinite weakly multiplicative system.*

Let $\{\Phi_n(x)\}$ be a bounded infinite orthogonal system on the set E. The expansion coefficients of every L^2_μ-integrable function tend to zero, hence there exists an index n_1 such that

$$\left|\int_E \Phi_{n_1}(x)\,d\mu(x)\right|\leqq\frac{1}{2^2}.$$

Set $\varphi_1(x)=\Phi_{n_1}(x)$. Suppose, we have chosen the functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_{n-1}(x)$ from $\{\Phi_n(x)\}$ in such a way that for every product $\varphi_{\nu_1}\varphi_{\nu_2}\dots\varphi_{\nu_k}$ with indices $\nu_1<\nu_2<\dots<\nu_k\leqq n-1$

$$\left|\int_E\prod_{j=1}^{k}\varphi_{\nu_j}(x)\,d\mu(x)\right|\leqq\frac{1}{2^{2(n-1)}}$$

holds true. All the finite products of the φ_k's being L^2_μ-integrable, for every product $\varphi_{\nu_1}\varphi_{\nu_2}\dots\varphi_{\nu_k}$ there exists a number n_m, depending on the choice of the product, such that

$$\left|\int_E\Phi_n(x)\prod_{j=1}^{k}\varphi_{\nu_j}(x)\,d\mu(x)\right|\leqq\frac{1}{2^{2n}}$$

for every $n\geqq n_m$. There are 2^{n-1} different products of this form, hence at most 2^{n-1} indices n_m. Denote by n_N the greatest of them and set $\varphi_n(x)=\Phi_{n_N}(x)$. Then

$$\left|\int_E\varphi_n(x)\prod_{j=1}^{k}\varphi_{\nu_j}(x)\,d\mu(x)\right|\leqq\frac{1}{2^{2n}}.$$

In this way we defined the infinite system $\{\varphi_n(x)\}$ by induction. To see that this system is weakly multiplicative, form all different products $\varphi_{\nu_1}\varphi_{\nu_2}\dots\varphi_{\nu_k}$ of the first n functions $\varphi_1, \varphi_2, \dots, \varphi_n$. There are 2^n such products, hence

$$\sum \Big| \int_E \varphi_{\nu_1}(x)\,\varphi_{\nu_2}(x)\dots\varphi_{\nu_k}(x)\,d\mu(x)\Big| \leqq \frac{1}{2^n} \tag{9}$$

where the sum has to be taken over all 2^n different products. The sum S of the absolute values of integrals of all possible finite products formed with the functions of the system $\{\varphi_n(x)\}$ is less than the sum of the sums (9) taken for $n=1, 2, \dots$. Hence

$$S < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

This estimate means just that $\{\varphi_n(x)\}$ is weakly multiplicative.

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