Structure of operators with numerical radius one

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Dedicated to Professor Béla Sz.-Nagy on his sixtieth birthday

1. Introduction

The *numerical radius* $w(T)$ of a bounded linear operator T on a Hilbert space $\mathfrak{H}$ is defined by

$$w(T) = \sup\{|(Th, h)| : \|h\| \leqq 1\}.$$

Important characterization of operators with numerical radius not greater than one was discovered by BERGER and subsequently generalized by SZ.-NAGY and FOIAŞ (see [1] I-1): $w(T) \leqq 1$ if and only if there is a unitary operator W on a Hilbert space $\mathfrak{K}$, containing $\mathfrak{H}$ as a subspace, such that for all $h \in \mathfrak{H}$ and $n \geqq 1$

$$T^n h = 2PW^n h$$

where P is the projection from $\mathfrak{K}$ to $\mathfrak{H}$. W is called a *unitary* 2-*dilation* of T.

The key result of the present paper is an intrinsic characterization of operators with numerical radius not greater than one (Theorem 1): $w(T) \leqq 1$ if and only if there are a selfadjoint contraction A and a contraction B such that $T = (1+A)^{\frac{1}{2}} B (1-A)^{\frac{1}{2}}$. This factorization theorem makes it possible to construct a unitary 2-dilation in simple matricial form just as the Schäffer description of a unitary dilation of a contraction (Theorem 2).

2. Factorization

$\mathfrak{H}$ is a Hilbert space, and $\bigoplus_{j=-\infty}^{\infty} \mathfrak{H}_j$ denotes direct sum of copies of $\mathfrak{H}$ in which $\mathfrak{H}$ is identified with $\cdots \oplus 0 \oplus \mathfrak{H}_0 \oplus 0 \oplus \cdots$ in the canonical way. A bounded linear operator S on $\bigoplus_j \mathfrak{H}_j$ can be represented by its matricial components, $[S_{j,k}]$, where $S_{j,k}$ is an operator on $\mathfrak{H}$, considered as an operator from $\mathfrak{H}_k$ to $\mathfrak{H}_j$.

Lemma 1. *If $w(T) \leqq 1$, there is a positive contraction X such that*

$$(Xh, h) = \inf_g \left(\begin{bmatrix} 1 & \frac{1}{2}T^* \\ \frac{1}{2}T & X \end{bmatrix} \begin{bmatrix} h \\ g \end{bmatrix}, \begin{bmatrix} h \\ g \end{bmatrix} \right). \tag{1}$$

Moreover X is the maximum of all positive contractions Y for which $\begin{bmatrix} 1-Y & \frac{1}{2}T^* \\ \frac{1}{2}T & Y \end{bmatrix}$ *is positive.*

Proof. Suppose that W is a unitary 2-dilation of T on a Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$. Let $X_0 = 1$ and X_n be the compression of $1-Q_n$ to $\mathfrak{H}$ where Q_n is the projection to $\bigvee_{j=1}^{n} W^{*j}(\mathfrak{H})$. This means that

$$(X_n h, h) = \inf_{h_1, \ldots, h_n \in \mathfrak{H}} \Big\| h + \sum_{j=1}^{n} W^{*j} h_j \Big\|^2 =$$

$$= \inf_{h_1, \ldots, h_n} \Big\{ (h, h) + \Big(h, \sum_{j=1}^{n} W^{*j} h_j\Big) + \Big(\sum_{j=1}^{n} W^{*j} h_j, h\Big) + \sum_{j,k=1}^{n} (W^{*j} h_j, W^{*k} h_k) \Big\} =$$

$$= \inf_{h_1, \ldots, h_n} \left(\begin{bmatrix} 1 & \frac{1}{2}T^* & \cdots & \frac{1}{2}T^{*n} \\ \frac{1}{2}T & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & 1 & \frac{1}{2}T^* \\ \frac{1}{2}T^n & \cdots & \frac{1}{2}T & 1 \end{bmatrix} \begin{bmatrix} h \\ h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} h \\ h_1 \\ \vdots \\ h_n \end{bmatrix} \right).$$

Since

$$\begin{bmatrix} 1 & \frac{1}{2}T^* & \cdots & \frac{1}{2}T^{*n} \\ \frac{1}{2}T & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & 1 & \frac{1}{2}T^* \\ \frac{1}{2}T^n & \cdots & \frac{1}{2}T & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & T^* & \cdots & T^{*n} \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & 1 & T^* \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}T^* & 0 \cdots & 0 \\ & \ddots & \ddots & \vdots \\ -\frac{1}{2}T & 1 & \ddots & 0 \\ & \ddots & \ddots & \\ 0 & \ddots & 1 & -\frac{1}{2}T^* \\ \vdots & \ddots & \ddots & \\ 0 & \cdots 0 & -\frac{1}{2}T & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ T & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & 1 & 0 \\ T^n & \cdots & T & 1 \end{bmatrix},$$

X_n admits the alternative representation

$$(X_n h, h) = \inf_{h_1, \ldots, h_n} \left(\begin{bmatrix} 1 & \frac{1}{2}T^* & 0 \cdots & 0 \\ & \ddots & \ddots & \vdots \\ \frac{1}{2}T & 1 & \ddots & 0 \\ & \ddots & \ddots & \\ 0 & \ddots & \ddots & \frac{1}{2}T^* \\ \vdots & \ddots & \ddots & \\ 0 & \cdots 0 & \frac{1}{2}T & 1 \end{bmatrix} \begin{bmatrix} h \\ h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} h \\ h_1 \\ \vdots \\ h_n \end{bmatrix} \right).$$

The corresponding representation for X_{n-1} yields

(2) $$(X_n h, h) = \inf_g \left(\begin{bmatrix} 1 & \frac{1}{2}T^* \\ \frac{1}{2}T & X_{n-1} \end{bmatrix} \begin{bmatrix} h \\ g \end{bmatrix}, \begin{bmatrix} h \\ g \end{bmatrix} \right).$$

Since X_n converges decreasingly to some X, transfer to limit in (2) leads to the relation (1).

Now if $\begin{bmatrix} 1-Y & \frac{1}{2}T^* \\ \frac{1}{2}T & Y \end{bmatrix}$ is positive, $X_{n-1} \geqq Y$ implies by (1) and (2)

$$(X_n h, h) \geqq \inf_g \left(\begin{bmatrix} 1 & \frac{1}{2}T^* \\ \frac{1}{2}T & Y \end{bmatrix} \begin{bmatrix} h \\ g \end{bmatrix}, \begin{bmatrix} h \\ g \end{bmatrix} \right) \geqq (Yh, h),$$

hence $X_n \geqq Y$. Now $X \geqq Y$ follows from $X_0 = 1 \geqq Y$.

Theorem 1. *The numerical radius of T is not greater than one if and only if T admits a factorization*

$$T = (1+A)^{\frac{1}{2}} B (1-A)^{\frac{1}{2}}$$

with a selfadjoint contraction A and a contraction B. Moreover in the set of such A there exist the maximum $A_{\max}$ and the minimum $A_{\min}$ and the corresponding $B_{\max}$ (resp. *the adjoint of $B_{\min}$*) *is isometric on the range of* $1 - A_{\max}$ (resp. *that of* $1 + A_{\min}$).

Proof. Suppose that T admits the factorization. Then

$$|(Th, h)| = |(B(1-A)^{\frac{1}{2}}h, (1+A)^{\frac{1}{2}}h)| \leqq \tfrac{1}{2}\{\|(1-A)^{\frac{1}{2}}h\|^2 + \|(1+A)^{\frac{1}{2}}h\|^2\} = \|h\|^2,$$

which shows $w(T) \leqq 1$.

Conversely, if $w(T) \leqq 1$, by Lemma 1 there is a positive contraction X with (1). Since (1) is equivalent to

$$\|(1-X)^{\frac{1}{2}}h\| = \sup_g \frac{|(\frac{1}{2}Th, g)|}{\|X^{\frac{1}{2}}g\|}$$

with convention $0/0 = 0$, to each h there corresponds uniquely f in the closure of the range of X such that

$$X^{\frac{1}{2}}f = \tfrac{1}{2}Th \quad \text{and} \quad \|f\| = \|(1-X)^{\frac{1}{2}}h\|.$$

Thus there is a contraction $B_{\max}$ which is isometric on the range of $1-X$ and $\frac{1}{2}T = X^{\frac{1}{2}} B_{\max} (1-X)^{\frac{1}{2}}$. Now $A_{\max} = 2X - 1$ meets the requirement. Given any factorization with A and B it follows with $Y = \frac{1}{2}(1+A)$ that

$$\begin{bmatrix} 1-Y & \frac{1}{2}T^* \\ \frac{1}{2}T & Y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1-A & T^* \\ T & 1+A \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} (1-A)^{\frac{1}{2}} & 0 \\ 0 & (1+A)^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & B^* \\ B & 1 \end{bmatrix} \begin{bmatrix} (1-A)^{\frac{1}{2}} & 0 \\ 0 & (1+A)^{\frac{1}{2}} \end{bmatrix}.$$

Since $\|B\|\leqq 1$ is equivalent to $\begin{bmatrix} 1 & B^* \\ B & 1 \end{bmatrix} \geqq 0$, $\begin{bmatrix} 1-Y & \frac{1}{2}T^* \\ \frac{1}{2}T & Y \end{bmatrix}$ is positive. Then Lemma 1 shows $Y\leqq X$, hence $A\leqq A_{\max}$. The minimum operator $A_{\min}$ can be obtained so as $-A_{\min}$ is the maximum operator for T^*. This completes the proof.

In general $A_{\max}$ is different from $A_{\min}$. This is shown with the simple 2×2 matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. In this case

$$A_{\max} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_{\min} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

In terms of the unitary 2-dilation W the maximum and the minimum operators are given by the following formulas:

$$A_{\max} = 1-2PQ_-P \quad \text{and} \quad A_{\min} = 2PQ_+P-1$$

where P is the projection to $\mathfrak{H}$ and Q_+(resp. Q_-) is the projection to $\bigvee_{n=1}^{\infty} W^n(\mathfrak{H})$ $\left(\text{resp. } \bigvee_{n=1}^{\infty} W^{-n}(\mathfrak{H})\right)$.

Theorem 2. *The numerical radius of T is not greater than one if and only if there is a contraction C such that*

$$T = 2(1-C^*C)^{\frac{1}{2}}C. \tag{3}$$

Under such factorization a unitary 2-dilation W is realized as an operator on $\bigoplus_{j=-\infty}^{\infty} \mathfrak{H}_j$ with components:

$$W_{k+1,k} = 1 \quad \textit{for} \quad k\geqq 1 \quad \textit{or} \quad k\leqq -3,$$

$$W_{-1,-2} = (1-CC^*)^{\frac{1}{2}}, \quad W_{-1,-1} = C(1-CC^*)^{\frac{1}{2}}, \quad W_{-1,0} = C^2,$$

$$W_{0,-2} = -C^*, \quad W_{0,-1} = (1-C^*C)^{\frac{1}{2}}(1-CC^*)^{\frac{1}{2}}, \quad W_{0,0} = (1-C^*C)^{\frac{1}{2}}C,$$

$$W_{1,-1} = -C^*, \quad W_{1,0} = (1-C^*C)^{\frac{1}{2}}$$

and $W_{j,k}=0$ for other j, k.

Proof. If $w(T)\leqq 1$, by Theorem 1

$$T = (1+A_{\max})^{\frac{1}{2}}B_{\max}(1-A_{\max})^{\frac{1}{2}}$$

and $B_{\max}$ is isometric on the range of $1-A_{\max}$. Let $C=2^{-\frac{1}{2}}B_{\max}(1-A_{\max})^{\frac{1}{2}}$. Then

$$1-C^*C = 1-\tfrac{1}{2}(1-A_{\max}) = \tfrac{1}{2}(1+A_{\max});$$

hence

$$T = 2(1-C^*C)^{\frac{1}{2}}C.$$

Suppose conversely that T admits a factorization (3). It is well known (see [1]

I-5) that a unitary dilation U of C is realized as the operator on $\bigoplus_{j=-\infty}^{\infty} \mathfrak{H}_j$ with components

$$U_{k+1,k} = 1 \quad \text{for} \quad k \geqq 1 \quad \text{or} \quad k \geqq -2,$$
$$U_{0,0} = C, \qquad U_{0,-1} = (1-CC^*)^{\frac{1}{2}},$$
$$U_{1,0} = (1-C^*C)^{\frac{1}{2}}, \qquad U_{1,-1} = -C^*,$$

and $U_{j,k}=0$ for other j, k. Then W in the assertion is written in the form $W=VU^2$ where V is the backward shift, that is,

$$V_{j,k} = \delta_{j,k-1} \quad \text{for all} \quad j,\ k;$$

hence W is unitary. W is a unitary 2-dilation of T if

$$(W^n)_{0,0} = \tfrac{1}{2}T^n \qquad (n = 1, 2, \ldots). \tag{4}$$

To prove (4) by induction, assume that

$$(W^n)_{-k,0} = 0 \quad (k \geqq 2), \quad (W^n)_{-1,0} = C^2T^{n-1} \quad \text{and} \quad (W^n)_{0,0} = \tfrac{1}{2}T^n,$$

which is valid for $n=1$ by definition. Matrix multiplication shows

$$(W^{n+1})_{-k,0} = (W^n)_{-k-1,0} = 0 \quad (k \geqq 2),$$

$$(W^{n+1})_{-1,0} = C(1-CC^*)^{\frac{1}{2}}(W^n)_{-1,0} + C^2(W^n)_{0,0} =$$
$$= C(1-CC^*)^{\frac{1}{2}}C^2T^{n-1} + \tfrac{1}{2}C^2T^n = \tfrac{1}{2}C^2T^n + \tfrac{1}{2}C^2T^n = C^2T^n$$

and

$$(W^{n+1})_{0,0} = (1-C^*C)^{\frac{1}{2}}(1-CC^*)^{\frac{1}{2}}(W^n)_{-1,0} + \tfrac{1}{2}T(W^n)_{0,0} =$$
$$= (1-C^*C)^{\frac{1}{2}}(1-CC^*)^{\frac{1}{2}}C^2T^{n-1} + \tfrac{1}{4}T^{n+1}$$
$$= \tfrac{1}{4}T^{n+1} + \tfrac{1}{4}T^{n+1} = \tfrac{1}{2}T^{n+1}.$$

Here, besides the relation $(1-C^*C)^{\frac{1}{2}}C = \frac{1}{2}T$, the well-known formula (see [1] I-5)

$$(1-CC^*)^{\frac{1}{2}}C = C(1-C^*C)^{\frac{1}{2}}$$

is used. This completes the proof.

Reference

[1] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space,* North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

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