Ergodic theory and the measure of sets in the Bohr group

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To Professor Béla Szőkefalvi-Nagy in honor of his 60th birthday

Introduction. Let U be a strongly continuous unitary representation of a locally compact group G on a Hilbert space H. If μ is a probability measure on G, $\int U_g\,d\mu$ is defined weakly so that $(\int U_g f\,d\mu, h)=\int (U_g f, h)\,d\mu$ for all h in H. For notation and terminology see [3].

Proposition 1. *If G is compact and μ is Haar measure then $\int U_g f\,d\mu = Pf$, where P is the orthogonal projection on the space $K=\{k\,|\,U_g k=k,\ U_g k=k,\ g\in G\}$.*

Proof. $\int U_g f\,d\mu$ is invariant since

$$(U_{g_0}\int U_g f d\mu, h) = (\int U_g f d\mu, U_{g_0}^* h) = \int (U_{g_0 g} f, h) = \int (U_g f, h)\,d\mu = (\int U_g f d\mu, h)$$

by the invariance of Haar measure. Since this holds for all $h\in H$, $U_{g_0}\int U_g f\,d\mu = \int U_g f\,d\mu$.

To complete the proof it must be shown that if $k\in K$ then $f-\int U_g f\,d\mu \perp k$. But

$$(f-\int U_g f d\mu, k) = (f,k)-\int (f, U_{g^{-1}}k)\,d\mu = (f,k)-\int (f,k)\,d\mu = 0.$$

Let μ_n be a sequence of probability measures on a locally compact Abelian group G. In BLUM and EISENBERG [2] the following theorem and corollary are proved.

Theorem. *The following are equivalent:*

(i) *For every continuous unitary representation U of G and every f in H, $\int U_g f\,d\mu_n$ converges in mean to Pf.*

(ii) *For every character x on G except that identically 1 the Fourier transforms $\hat\mu_n(x)=\int \langle x, g\rangle\,d\mu_n$ converge to 0.*

(iii) *μ_n considered as restrictions of measures on the Bohr compactification $\bar G$ of G converge weakly to Haar measure on $\bar G$.*

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(iv) *For every character x of infinite order the measures μ_n^x induced by x on the unit circle in the complex plane converge weakly to normalized Lebesgue measure on the circle and for every character x of order m, $m=0, 1, 2, \dots$ the measures μ_n^x converge weakly to Haar measure on the m^{th} roots of unity.* (The measure μ_n^x induced by x is defined so that $\mu_n^x(B)=\mu_n(\{g|(x, g)\in B\})$.)

Corollary. *If E_n is a sequence of sets in G such that for all g in G, $\frac{\mu(E_n\cap gE_n)}{\mu(E_n)}\to 1$, where μ is Haar measure, then the measures $\mu_n(A)=\frac{\mu(A\cap E_n)}{\mu(E_n)}$ converge weakly to Haar measure on $\bar{G}$.*

Proposition 1 and the theorem lead to the questions studied in this paper. Question one asks, does $\int_{\bar{G}}\overline{(U_g f, h)}\,d\mu=(Pf, h)$ hold, where $\overline{(U_g f, h)}$ is a suitable extension of $(U_g f, h)$ to $\bar{G}$ and μ is Haar measure on $\bar{G}$? The theorem says that $(Pf, h)=\lim_n \int (U_g f, h)\,d\mu_n$, where μ_n converges weakly to Haar measure on $\bar{G}$ and Proposition 1 says that the statement is true when $\bar{G}$ is already compact. If the answer is yes, there would be an interesting expression for Pf in terms of the action of U_g on f.

Question two asks for which sequences of integers the mean ergodic theorem holds; i.e., when is it true that $\frac{1}{N}\sum_1^N T^{n_k}f$ converges in mean to the projection of f on the space of elements invariant under T for every unitary T. The theorem says that a sequence is ergodic if and only if the probability measure μ_N giving measure $\frac{1}{N}$ to each integer $n_1, n_2, \dots, n_N$ converges weakly to Haar measure on the Bohr compactification of the integers.

Both questions relate to the study of the measure of sets in the Bohr compactification of the integers.

1. This section is concerned with the first question. It is seen that properties of the spectral resolution of U are crucial.

Proposition 2. *If U has pure point spectrum then $(Pf, h)=\int_{\bar{G}}\overline{(U_g f, h)}\,d\mu$, where $\overline{(U_g f, h)}$ is the unique continuous extension of $(U_g f, h)$ to $\bar{G}$ and μ is Haar measure on $\bar{G}$.*

Proof. By Stone's theorem and the assumption on discrete spectrum $(U_g f, h)= =\int_{\hat{G}}\langle x, g\rangle\,d(E_x f, h)=\sum C_k\langle x_k, g\rangle$, where $\hat{G}$ is the dual group of G and $\sum|C_k|<\infty$. Since $|\langle x_k, g\rangle|=1$, $(U_g f, h)$ is a uniform limit of almost periodic functions and hence is almost periodic. It follows that $(U_g f, h)$ is the restriction of a continuous function on $\bar{G}$. By the theorem

$$(Pf, h) = \lim \int_G (U_g f, h)\, d\mu_n = \lim \int_{\bar{G}} \overline{(U_g f, h)}\, d\mu_n = \int_{\bar{G}} \overline{(U_g f, h)}\, d\mu.$$

In fact, if a function $\varphi(g)$ is merely bounded and continuous a.e. $d\mu$ on $\bar{G}$,

$$\int \varphi(g)\, d\mu_n \to \int \varphi(g)\, d\mu.$$

Lemma. *Let X be a normal topological space and μ a finite regular measure on X. If $\mu_n \to \mu$ weakly and if φ is bounded and continuous a.e. $d\mu$, then $\int \varphi\, d\mu_n \to \int \varphi\, d\mu$.*

As the proof of this lemma is somewhat technical and independent of the rest of the paper it will be relegated to an appendix.

For the remainder of the paper attention is limited to unitary groups generated by a single operator.

Proposition 3. *If the maximal spectral type of T has no continuous singular part then*

$$\int_{\bar{Z}} \overline{(T^n f, h)}\, d\mu = (Pf, h),$$

where $\overline{(T^n f, h)}$ is a continuous a.e. $d\mu$ extension of $(T^n f, h)$.

Proof.

$$(T^n f, h) = \int_0^{2\pi} e^{int}\, d(E_t f, h) = \int_0^{2\pi} e^{int} \varrho(t)\, dt + \Sigma C_k e^{int_k},$$

where $\int_0^{2\pi} |\varrho(t)|\, dt < \infty$ and $\Sigma |C_k| < \infty$.

As in Proposition 2, $\sum C_k e^{int_k}$ has a unique continuous extension to $\bar{Z}$. By the Riemann—Lebesgue lemma $\int e^{int_k} \varrho(t)\, dt \to 0$ as $n \to \infty$. It follows that if

$$\overline{(T^n f, h)} = \begin{cases} (T^n f, h) & \text{on } Z, \\ \overline{\Sigma C_k e^{int_k}} & \text{on } \bar{Z} - Z, \end{cases}$$

then $\overline{(T^n f, h)}$ is continuous except on Z itself. That is, if $n_k \to g \in \bar{Z} - Z$ then $\int e^{in_k t} \varrho(t) \to 0$ so that $(T^{n_k} f, h) \to \overline{\sum C_k e^{int_k}(g)}$. But Z has measure 0 in $\bar{Z}$ so that $\overline{(T^n f, h)}$ is continuous a.e. $d\mu$. By the theorem and lemma,

$$(Pf, h) = \lim_k \int (T^n f, h)\, d\mu_k = \lim_k \int \overline{(T^n f, h)}\, d\mu_k = \int \overline{(T^n f, h)}\, d\mu.$$

Finally T with continuous singular spectrum must be considered. It is no longer true that $(T^n f, h) \to 0$. However, $(T^n f, h)$ does approach 0 except on a sequence of density 0. An increasing sequence n_k has density 0 if $\frac{\#\{n_k<n\}}{n}\to 0$. (If A is a set then $\#(A)$ is the cardinality of A.) If it could be shown that such sequences have closure of Haar measure 0 in $\bar{Z}$ then by a similar argument to that in Proposition 3 it could be shown that $\int \overline{(T^n f, h)}\,d\mu=(Pf, h)$, were $\overline{(T^n f, h)}$ is defined as in Proposition 3.

This leads to the problem of determining when the Haar measure of the closure of sets of integers in $\bar{Z}$ is zero.

2. Question one leads to the question of which sequences have closure of measure 0 in $\bar{Z}$. Question two asks which sequences induce measures converging weakly to Haar measure on $\bar{Z}$. Such scquences must be dense in $\bar{Z}$. Otherwise there is an open set Θ in $\bar{Z}$ containing no elements of the sequence. By Urysohn's lemma there is a non-negative continuous function φ with support inside Θ such that $\int\varphi\,d\mu>0$. But $\int\varphi\,d\mu_n\equiv 0$. It will thus be of interest to find conditions merely for denseness of sequences of integers in $\bar{Z}$.

Proposition 4. *Cosets of the subgroup $H=\{0, \pm m, \pm 2m, \dots\}$ have disjoint closures in $\bar{Z}$ and each has measure $1/m$.*

Proof. A neighborhood of g_0 in $\bar{Z}$ is defined by $\{g\,|\,|\langle t_i, g\rangle-\langle t_i, g_0\rangle|<\varepsilon\}$ where $\varepsilon>0$ and $0\leqq t_i<2\pi$. Consider the character corresponding to $t=\frac{2\pi}{m}$. Then if $g\in k+H$, $\langle t, g\rangle=e^{2\pi ik/m}$ while if $g'\in k'+H$, $\langle t, g'\rangle=e^{2\pi ik'/m}$. If $g_0\in\overline{k+H}$ then $\langle t, g_0\rangle= =e^{2\pi ik/m}$ while if $g_0\in\overline{k'+H}$, $\langle t, g_0\rangle=e^{2\pi ik'/m}$. Since $\bigcup_{k=1}^{m}(k+H)$ is dense in $\bar{Z}$ and $\overline{k+H}$ and $\overline{k'+H}$ are translates of one another, $\mu(\overline{k+H})=1/m$, $k=1,2,\dots,m$.

Corollary. *The following sequences have closure of measure* 0 *in* $\bar{Z}$.

(i) $n!$,

(ii) a^n, *where a is an integer,*

(iii) p_n, *the sequence of primes,*

(iv) n^k, *where k is a fixed integer* $\geqq 2$.

Proof. Since each integer has measure 0 and the topology is Hausdorff, a finite number of elements in the sequence can be neglected.

(i) For $n\geqq m$, $n!\equiv 0 \mod m$. Hence $\mu(\overline{\{n!\}})\leqq 1/m$ since $\{n!\,|\,n\geqq m\}$ is a subset of $\{0, \pm m, \pm 2m, \dots\}$. But m is arbitrary. Hence $\mu(\overline{\{n!\}})=0$.

(ii) For $n\geqq m$, $a^n\equiv 0 \mod a^m$. Thus $\mu(\overline{\{a^n\}})\leqq 1/a^m$. Again m is arbitrary so $\mu(\overline{\{a^n\}})=0$.

(iii) Consider the set of residues of all primes modulo m. For a given prime p either $p\leqq m$ or $p=km+r$, where $k\geqq 1$. In the latter case r is relatively prime to m. Otherwise p would be divisible by the greatest common divisor of m and r. By the prime number theorem the number of primes less than or equal to m divided by m goes to 0. The number of integer r less than m and relatively prime to m is just the Euler Φ function of m. $\Phi(m) = m \prod_i \left(1-\frac{1}{q_i}\right)$, where $m=q_1^{\alpha_1}\dots q_s^{\alpha_s}$ is the prime factorization of m. $\frac{\Phi(m)}{m} = \prod_i \left(1-\frac{1}{q_i}\right)$ which can be made arbitrarily small since $\sum 1/p_i=\infty$, where p_i is the sequence of primes (LEVEQUE [5], p. 100). Thus $\mu(\overline{\{p_i\}}) \leqq \frac{\Phi(m)}{m}+o(m)$. Since m can be any product of primes, $\mu(\overline{\{p_i\}})=0$.

(iv) By Dirichlet's Theorem (LEVEQUE [5], p. 76) there is an infinite number of primes of the form $p = kn+1$ as n goes through the integers.

For such primes p, $p-1$ is divisible by k. As a consequence of Theorem 4-14 (LEVEQUE [4], p. 58) there are $\frac{p-1}{k}+1$ residue classes occupied by the residues of kth powers mod p. The fraction of classes occupied is $\frac{1}{p}\left(\frac{p-1}{k}+1\right) \leqq \frac{1}{k}+\frac{1}{p} \leqq \frac{5}{6}$ if $k\geqq 2$ and $p>3$. For a fixed k choose an infinite sequence of primes $p_m>0$ of the above form. By the Chinese Remainder Theorem

$$Z_{p_1 p_2 \dots p_m} \cong Z_{p_1}\times Z_{p_2}\times\dots\times Z_{p_m}$$

via the map $x\to(x \bmod p_1, x \bmod p_2, \dots, x \bmod p_m)$.

Thus the number of residue classes occupied by kth powers modulo $p_1\dots p_m$ is $\prod_{i=1}^{m}\left(\frac{p_m-1}{k}+1\right)$. The fraction occupied is less than $\left(\frac{5}{6}\right)^m$ which can be made arbitrarily small by choosing m large enough. Hence $\mu(\overline{\{n^k\}})=0$.

If all sequences of density 0 had closures of measure 0 in $\bar{Z}$ question one would be answered. Unfortunately this is not the case as is shown in the next proposition. The question is still open as to whether the sequence where the Fourier transform of a continuous singular measure fails to go to zero can be of this type, namely, have closure of positive measure.

Proposition 5. *Consider a set S of integers of the form C_n+k, $k = 0, 1, 2, \dots$ $\dots, n-1$, where C_n increases and $\frac{n(n+1)}{C_n}\to 0$. As a sequence, S has density 0 but $\bar{S}=\bar{Z}$.*

Proof. The number of elements of S less than C_n is $\frac{n(n+1)}{2}$. Thus on the subsequence C_n

$$\frac{\#\{\text{elemens of } S \text{ which are less than } C_n\}}{C_n} = \frac{n(n+1)}{2C_n}\to 0.$$

Since there are relatively few terms of the sequence between C_n and C_{n+1}, if n is large enough the oscillation in the density between C_n and C_{n+1} goes to zero. Hence the sequence of elements of S has density 0.

However, it is easy to check that the sequence of sets E_n of the first n elements of S satisfies the conditions of the corollary to the theorem in the introduction. That is, for any k, $\frac{\#(E_n \cap E_n + k)}{n} \to 1$. (The fraction of elements of E_n with a kth successor approaches one.) By the corollary the measures $\mu_n(A) = \frac{\#(A \cap E_n)}{n}$ converge weakly to Haar measure on $\bar{Z}$ and by the argument at the beginning of this section, S must be dense in $\bar{Z}$.

Corollary. *Mean ergodic theorems hold for some sequences of density* 0.

Proof. This follows from the theorem and the proof of this proposition.

The sequence described in Proposition 5 is not only dense in $\bar{Z}$, its induced sequence of measures converges weakly to Haar measure on $\bar{Z}$. The remainder of the paper considers conditions for denseness alone.

For this part of the work a generalized Kronecker Theorem is needed. As stated in RUDIN [6], p. 98, G is a locally compact Abelian group. For $x \in G$, put $S(x) = T$ if x has infinite order; if x has order q, put $S(x) =$ the qth roots of unity.

Theorem. *Suppose E is a finite independent (in the group theoretic sense) set in G, f is a function on E such that $f(x) \in S(x)$ for all $x \in E$ and $\varepsilon > 0$. Then there exists a $\gamma \in \Gamma$ such that*

$$|\langle x, \gamma \rangle - f(x)| < \varepsilon \quad (x \in E).$$

A concrete Kronecker theorem is in KATZNELSON [4], p. 60.

For our purposes G is the unit circle in C and $\Gamma = Z$. x has infinite order if it is of the form $2\pi\alpha$, where α is irrational and x has finite order if x is of the form $2\pi \frac{k}{m}$. The abstract Kronecker theorem gives a necessary condition for denseness in the Bohr group. Namely,

Corollary. *In order that a sequence of integers be dense in $\bar{Z}$ it is necessary that for every finite independent set E in T, and every f such that $f(x) \in S(x)$, and $\varepsilon > 0$ there exist an n_k in the sequence such that $|f(x) - e^{in_k x}| < \varepsilon$ for $x \in E$.*

Proof. There is some integer n such that $|f(x) - e^{inx}| < \varepsilon/2$ for $x \in E$. To approximate this integer n in the topology on the Bohr group there must be an n_k such that $|e^{inx} - e^{in_k x}| < \varepsilon/2$ for $x \in E$. By the triangle inequality $|f(x) - e^{in_k x}| < \varepsilon$ for $x \in E$.

It also follows from the Kronecker theorem that covering every residue class of every integer is not sufficient for density in $\bar{Z}$.

Corollary. *There exists a sequence n_k with elements in each residue class of every integer with $\mu(\{\overline{n_k}\})=0$, where μ is Haar measure on $\bar{Z}$.*

Proof. Take a fixed irrational number α and an arbitrary integer m. By the Kronecker theorem, given any residue class j of m and any $\varepsilon>0$ there is an integer n with $e^{\frac{2\pi in}{m}}=e^{\frac{2\pi ij}{m}}$ and $|e^{2\pi i\alpha n}-1|<\varepsilon$. By varying j, m and ε a sequence n_k can be selected going through every residue class of every integer with $e^{2\pi i\alpha n_k}$ converging to 1.

Such a sequence $\{n_k\}$ must have closure of measure 0. To see this note that if S is a set of integers such that $|e^{2\pi i\alpha n}-1|<\varepsilon$ for $n\in S$, then for any g in $\bar{S}$, $|\langle 2\pi\alpha, g\rangle-1|<\varepsilon$. Since $e^{2\pi i\alpha k}$ is dense in the circle as k goes through the integers there exist $\left[\frac{2\pi}{\varepsilon}\right]$ translates of S with disjoint closures, where $\left[\frac{2\pi}{\varepsilon}\right]$ is the greatest integer less than $\frac{2\pi}{\varepsilon}$. Hence by the same argument as Proposition 4 $\mu(\bar{S})\leqq 1/\left[\frac{2\pi}{\varepsilon}\right]$.

For the constructed sequence n_k we can neglect a finite number of terms to show $\mu(\{\overline{n_k}\})\leqq 1/\left[\frac{2\pi}{\varepsilon}\right]$ for all $\varepsilon>0$. Hence $\mu(\{\overline{n_k}\})=0$.

Let n_k be a sequence of integers and E_m the set of the first m of them. E_m+k is the shift of the set E_m by k. Let

(i) be the statement $\lim\limits_{m=\infty}\frac{\#\{E_m\cap E_m+k\}}{m}=1$ for all k.

(ii) be the statement $\mu_m(A)=\frac{\#(A\cap E_m)}{m}$ converges weakly to Haar measure on $\bar{Z}$.

(iii) be the statement that if $x=2\pi\alpha$, α irrational then e^{ixn_k} is uniformly distributed on the unit circle and if $x=2\pi r$, r a primitive qth root of unity then e^{ixn_k} is uniformly distributed on the qth roots of unity.

(iv) be the statement that the set $\{n_k\}$ is dense in $\bar{Z}$, and

(v) be the statement

If E is a finite independent set on the circle and $f(x)$ is a function on E of absolute value one such that if x is a primitive qth root of unity $f(x)$ is a qth roots of unity then $\forall\varepsilon>0$, there is an n_k such that

$$|f(x)-e^{ixn_k}|<\varepsilon.$$

Putting several results together we get

Theorem 1. (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v).

Proof. (i)$\Rightarrow$(ii) is in Blum—Eisenberg [2] and is stated in the introduction. (ii)$\Rightarrow$(iii) is part of the theorem in the introduction. (iii)$\Rightarrow$(iv) is from the argument at the beginning of section 2. (iv)$\Rightarrow$(v) is the corollary to the abstract Kronecker Theorem.

What is amazing about this result is that it says if the statement (i) is true then the sequence can be used to approximate functions in the Kronecker sense. (ii), (iii) and (iv) and (v) each seem very difficult to verify themselves, but (i) gives many sequences satisfying (ii) to (v). In fact, no sequences are known which satisfy (ii) but not (i).

Appendix

Lemma. *Let X be a normal topological space and μ a finite regular measure on X. If $\mu_n \to \mu$ weakly and if f is bounded and continuous a.e. $d\mu$, then $\int f d\mu_n \to \int f d\mu$.*

Proof. Take an open set Θ in X. There is a closed set $C \subset \Theta$ with $\mu(\Theta - C) \leqq C$. By Urysohn's lemma there is a continuous function r with $r=1$ on C, $|r| \leqq 1$, and $r=\Theta$ on Θ^C. $\mu_n(\Theta) \geqq \int r\, d\mu_n \to \int r\, d\mu \geqq \mu(\Theta) - \varepsilon$. Hence $\underline{\lim}\, \mu_n(\Theta) \geqq \mu(\Theta)$.

The set $A = \{a \mid \mu\{x \mid f(x)=a\} > 0\}$ is countable since μ is finite. Approximate $\int f d\mu_n$ and $\int f d\mu$ by $\sum a_i \mu_n\{a_i < f < b_i\}$ and $\sum a_i \mu\{a_i < f < b_i\}$, respectively, where $|a_i - b_i| < \varepsilon$ and a_i and b_i do not belong to the countable set A.

Let $\{a_i < f < b_i\} = C_i$. If $x \in \bar{C}_i - C_i$ then either $f(x) = a_i$ or b_i or x is a point of discontinuity of f. Thus $\mu(\bar{C}_i - C_i) = 0$.

Let $x \in C_i - C_i^0$. Then $f(x) \in (a_i, b_i)$ while for each neighborhood of x there is a y with $f(y) \notin (a_i, b_i)$. Hence x is a discontinuity point of f and $\mu(C_i - C_i^0) = 0$. Thus $\mu(\bar{C}_i - C_i^0) = 0$, and $\mu(\bar{C}_i) = \mu(C_i) = \mu(C_i^0)$. But

$$\overline{\lim}\, \mu_n(\bar{C}_i) \leqq \mu(\bar{C}_i) = \mu(C_i^0) \leqq \underline{\lim}\, \mu_n(C_i^0).$$

Thus $\lim \mu_n(C_i) = \mu(C_i)$.

Hence $\sum a_i \mu_n(C_i) \to \sum a_i \mu(C_i)$. Since $\int f\, d\mu_n \sim \sum a_i \mu_n(C_i)$ and $\int f\, d\mu \sim \sim \sum a_i \mu(C_i)$, it must be that $\int f\, d\mu_n \to \int f\, d\mu$.

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