On the Banach—Steinhaus theorem and approximation in locally convex spaces

By P. L. BUTZER, K. SCHERER and U. WESTPHAL in Aachen (FRG)

Dedicated to Professor B. Sz.-Nagy on his 60th birthday on July 29, 1973, in friendship and high esteem

1. Introduction

The Banach—Steinhaus theorem essentially states that a family of bounded operators is convergent on a whole space if and only if the operators are uniformly bounded as well as convergent on a dense subspace. It is the purpose of this note to extend part of the results of P. L. BUTZER—K. SCHERER [3], namely to give necessary and sufficient conditions upon a family of operators such that they tend to some limiting operator with a given order of approximation. This can be interpreted as the Banach—Steinhaus theorem equipped with a *rate* of convergence. The results are stated for locally convex spaces. They yield applications to weighted approximation, error estimates for quadrature formulae and the mean ergodic theorem. It is to be noted that all three applications are of a quite different structure.

2. The Banach—Steinhaus theorem with rate

Let X and Y be locally convex Hausdorff spaces with topologies generated by the families of filtrating seminorms $\{p\}$, $\{q\}$, respectively.

Let T_ϱ, $\varrho\geqq 0$, T be bounded mappings defined on X into Y such that $T_\varrho - T$ is sublinear for each $\varrho\geqq 0$, i.e.

$$q[(T_\varrho-T)(f_1+f_2)]\leqq q[(T_\varrho-T)f_1]+q[(T_\varrho-T)f_2]$$

$$(1)\qquad q[(T_\varrho-T)(af)]=q[a(T_\varrho-T)f]$$

for each $q\in\{q\}$ and $f_1, f_2, f\in X$, $a\in\mathbf{R}$. Provided X is barrelled, the theorem of Banach—Steinhaus states: the family $\{T_\varrho f;\ \varrho\geqq 0\}$ converges to Tf in the topology of Y for each $f \in X$, i.e. for each $q \in \{q\}$ one has

(2) $$\lim_{\varrho \to \infty} q[T_\varrho f - Tf] = 0 \qquad (\forall f \in X)$$

if and only if

(3, i) $\{T_\varrho;\ \varrho \geqq 0\}$ is uniformly bounded, i.e. to each $q \in \{q\}$ there exists $p \in \{p\}$ and a constant $M > 0$ such that

$$\sup_{\varrho \geqq 0} q[(T_\varrho - T)f] \leqq Mp(f) \qquad (\forall f \in X),$$

and

(3, ii) $\{T_\varrho f;\ \varrho \geqq 0\}$ converges to Tf in the topology of Y for each $f \in A$, A being a total set in X.

For the Banach—Steinhaus theorem, see H. G. Garnir—M. De Wilde—J. Schmets [8, p. 453], N. Bourbaki [1, p. 27], H. H. Schaefer [14, p. 86].

In order to study the rate of convergence of the given family, it is useful to introduce a quantity in place of the classical modulus of continuity, namely a modification of the K-functional. It is defined for $t > 0$, $f \in X$, $p \in \{p\}$ and $\bar{p} \in \{\bar{p}\}$ by

(4) $$K(t, f; X, A)_{p,\bar{p}} = \inf_{g \in A} \{p(f-g) + t\bar{p}(g)\},$$

where $(A, \{\bar{p}\})$ is a subspace of $(X, \{p\})$.

Theorem 1. *Let $(X, \{p\})$, $(A, \{\bar{p}\})$, $(Y, \{q\})$ be locally convex spaces with $A \subset X$. Let T_ϱ, $\varrho \geqq 0$, and T be bounded operators mapping X into Y such that $T_\varrho - T$ is sublinear for each $\varrho \geqq 0$. Then to each $q \in \{q\}$ there exist $p \in \{p\}$ and $\bar{p} \in \{\bar{p}\}$ such that*

(5) $$q[(T_\varrho - T)f] \leqq C\varphi(\varrho) K(\psi(\varrho)[\varphi(\varrho)]^{-1}, f; X, A)_{p,\bar{p}} \qquad (\forall f \in X),$$

where $\varphi(\varrho)$ and $\psi(\varrho)$ are positive functions of ϱ, if and only if

(6, i) $$q[(T_\varrho - T)f] \leqq M\varphi(\varrho)p(f) \qquad (\forall f \in X)$$

and

(6, ii) $$q[(T_\varrho - T)f] \leqq D\psi(\varrho)\bar{p}(f) \qquad (\forall f \in A),$$

where C, D and M are constants independent of ϱ and f.

Proof. To establish the implication (5)⇒(6), first note that

(7, i, ii) $$K(t, f; X, A)_{p,\bar{p}} \leqq \begin{cases} p(f) & \forall f \in X, \\ t\bar{p}(f) & \forall f \in A. \end{cases}$$

Then (5) implies by (7, i, ii) upon setting $t=\psi(\varrho)[\varphi(\varrho)]^{-1}$

$$q[(T_\varrho-T)f] \leqq \begin{cases} C\varphi(\varrho)p(f), & \forall f\in X \\ C\varphi(\varrho)\psi(\varrho)[\varphi(\varrho)]^{-1}\bar{p}(f), & \forall f\in A. \end{cases}$$

This yields (6, i) and (6, ii) with $M=D=C$.

To establish the converse, in view of the sublinearity of $T_\varrho-T$ one has by (6, i, ii) for each $g\in A$

$$q[(T_\varrho-T)f] \leqq q[(T_\varrho-T)(f-g)]+q[(T_\varrho-T)g] \leqq$$

$$\leqq M\varphi(\varrho)p(f-g)+D\psi(\varrho)\bar{p}(g) \leqq$$

$$\leqq \max(M,D)\varphi(\varrho)\{p(f-g)+\psi(\varrho)[\varphi(\varrho)]^{-1}\bar{p}(g)\}.$$

Taking the infimum over all $g\in A$ one has that for all $f\in X$

$$q[(T_\varrho-T)f] \leqq \max(M,D)\,\varphi(\varrho)\,K(\psi(\varrho)[\varphi(\varrho)]^{-1}, f; X, A)_{p,\bar{p}}.$$

This proves the theorem.

The sufficient direction of Theorem 1 in case X is a Banach space with $X=Y$, $T=I$, may be found in P. L. Butzer—K. Scherer [3]. In this case, for $\psi(\varrho)\to 0$ as $\varrho\to\infty$, condition (6, ii) is referred to as a Jackson-type inequality. In this respect note that P. O. Runck [13] has actually given necessary and sufficient conditions upon T_ϱ such that a Jackson-type inequality is satisfied.

In the foregoing theorem the constants C, D, M were independent of f and ϱ. In the following deeper and more theoretical version the corresponding constants C and D may depend upon the element f.

Theorem 2. *Let $(X, \{p\})$, $(A, \{\bar{p}\})$, $(Y, \{q\})$ be locally convex Hausdorff spaces such that A is continuously embedded in X, i.e. to each $p\in\{p\}$ there is $\bar{p}\in\{\bar{p}\}$ and $c>0$ with $p(f)\leqq c\bar{p}(f)$ for all $f\in A$. In addition, let X as well as A be barrelled. If T_ϱ, $\varrho\geqq 0$, and T are bounded operators mapping X into Y such that $T_\varrho-T$ is sublinear for each $\varrho\geqq 0$, then the following two assertions are equivalent:*

(8) *to each $q\in\{q\}$ there is $p\in\{p\}$ and $\bar{p}\in\{\bar{p}\}$ such that* $(\delta>0)$

$$q[(T_\varrho-T)f] = O[K(\varrho^{-\delta}, f; X, A)_{p,\bar{p}}] \quad (\forall f\in X),$$

(9) *to each $q\in\{q\}$ there is $p\in\{p\}$ and $M>0$ such that*

(9, i) $$\sup_{\varrho\to 0} q[T_\varrho-T)f] \leqq Mp(f) \quad (\forall f\in X).$$

(9, ii) $$q[(T_\varrho-T)f] = O(\varrho^{-\delta}) \quad (\forall f\in A).$$

Proof. (8)⇒(9): The estimate (8) together with (7, i, ii) implies ($t=\varrho^{-\delta}$)

$$q[(T_\varrho - T)f] = \begin{cases} O(1), & \forall f \in X \\ O(\varrho^{-\delta}), & \forall f \in A. \end{cases}$$

The second assertion is the required (9, ii). To obtain (9, i), apply the uniform boundedness theorem (= necessary condition of classical Banach—Steinhaus theorem) to $q[(T_\varrho - T)f] = O(1)$, all $f \in X$, noting that X is barrelled.

(9)⇒(8): A being barrelled, (9, ii) implies by the uniform boundedness principle that there exists $\bar{p} \in \{\bar{p}\}$ and $D > 0$ such that condition (6, ii) of Theorem 1 holds. (6, i) being valid here by assumption, one may therefore apply Thm. 1 with $\varphi(\varrho) = 1$ and $\psi(\varrho) = \varrho^{-\delta}$.

Concerning the structure of Theorem 2 in comparison with the Banach—Steinhaus theorem, the assertions (2), (3, ii) on convergence per se are replaced by the assertions (8), (9, ii) involving an order of convergence. Indeed, if A is dense in X $\lim_{t\to 0+} K(t, f; X, A)_{p,\bar{p}} = 0$. This is the situation in the applications to follow.

3. Weighted approximation

The first application will be concerned with weighted approximation; it will turn out to be an actual example of approximation in a locally convex space. Here the space Y will be seen to be equal to the locally convex space X and the limit operator will be the identity. The corresponding problem was first considered by J. Kemper—R. J. Nessel [10] using classical methods.

Let E be the space of functions given on the reals $\mathbf{R}$ which are either uniformly continuous and bounded on $\mathbf{R}$ or measurable and pth power ($1 \leqq p < \infty$) integrable on $\mathbf{R}$, and let E be normed in the usual fashion. Let E_{loc} be those functions which are either continuous on $\mathbf{R}$ or pth power integrable on each compact subset of $\mathbf{R}$.

Let

$$X = \{f \in E_{\text{loc}};\quad \|e^{-\beta x^2} f(x)\|_E < +\infty,\quad \forall \beta > 0\};$$

it is a locally convex Hausdorff space with respect to the family of norms

$$p_\beta(f) = \|e^{-\beta x^2} f(x)\|_E \qquad (\forall f \in E).$$

Let

$$A = \{f \in X\,; f, f' \text{ loc. abs. continuous}, f'' \in X\}$$

and

$$\bar{p}_\beta(f) = \|e^{-\beta x^2} f''(x)\|_E \qquad (\forall f \in A)$$

be a family of seminorms on A.

It is our purpose to consider the Weierstrass integral

$$(W_\varrho f)(x) = \frac{\varrho}{2\sqrt{\pi}} \int_{-\infty}^{\infty} f(x-u)\exp\left(-\frac{\varrho^2 u^2}{4}\right) du$$

for $\varrho > 0, f \in X$. This defines a family of operators W_ϱ on X into itself which converge in the topology of X towards the identity operator, i.e., for each $\beta > 0$ and each $f \in X$

$$\lim_{\varrho\to\infty} \|e^{-\beta x^2}[(W_\varrho f)(x) - f(x)]\|_E = 0.$$

Now $W_\varrho - I$ satisfies the hypotheses of Thm. 1. Indeed, taking $\varrho_0 > 0$ arbitrary fixed, it is easy to verify, using [10], that for each $\beta > 0$

$$\|e^{-\beta x^2}[(W_\varrho f)(x) - f(x)]\|_E \leqq (1+\sqrt{2})\|e^{-\eta x^2} f(x)\|_E \qquad (\varrho \geqq \varrho_0;\ \forall f \in X),$$

where $\eta = 1/2 \min(\beta, \varrho_0^2/8)$. Thus (6, i) is satisfied with $M = 1+\sqrt{2}$ and $\varphi(\varrho) = 1$. Likewise with (6, ii); indeed, for each $\beta > 0$ (see [10])

$$\|e^{-\beta x^2}[(W_\varrho f)(x) - f(x)]\|_E \leqq 4\sqrt{2}\,\varrho^{-2}\,\|e^{-\eta x^2} f''(x)\|_E \qquad (\varrho \geqq \varrho_0;\ f \in A).$$

Thus one may apply Thm. 1, (6, i, ii)$\Rightarrow$(5), to get for each $\beta > 0$

$$\|e^{-\beta x^2}[(W_\varrho f)(x) - f(x)]\|_E \leqq 4\sqrt{2}\, K(\varrho^{-2}, f; X, A)_{\eta,\eta}.$$

The following lemma is of importance (compare [2, p. 192] in the case of semigroup operators)

Lemma. *Under the preceding hypotheses we have*

$$K(t^2, f; X, A)_{\eta,\beta} \leqq \tfrac{3}{2}\,\omega_2(t, f; X)_\zeta \qquad (t > 0;\ f \in X),$$

where $\zeta = \min(\eta, \beta)$ *and*

$$\omega_2(t, f; X)_\zeta = \sup_{|s|\leqq t} \|e^{-\zeta x^2}[f(x+s) + f(x-s) - 2f(x)]\|_E.$$

Proof. It is obvious that

$$f(x) = -\frac{1}{2t^2} \int_{-t/2}^{t/2}\int_{-t/2}^{t/2} [f(x+\tau_1+\tau_2) + f(x-\tau_1-\tau_2) - 2f(x)]\, d\tau_1\, d\tau_2 + g_t(x)$$

where

$$g_t(x) = \frac{1}{t^2} \int_{-t/2}^{t/2}\int_{-t/2}^{t/2} f(x+\tau_1+\tau_2)\, d\tau_1\, d\tau_2.$$

This yields, first of all,

$$p_\eta(f - g_t) \leqq \tfrac{1}{2}\,\omega_2(t, f; X)_\eta.$$

Furthermore, since $g_t''(x) = t^{-2}[f(x+t)+f(x-t)-2f(x)]$, we have

$$\bar{p}_\beta(g_t) \leqq t^{-2}\omega^2(t,f;X)_\beta.$$

Combining the results, the desired inequality follows immediately.

Proposition 1. *For each $\beta>0$ there is $\eta(=1/2\min(\beta, \varrho_0^2/8))$ such that*

$$\|e^{-\beta x^2}[(W_\varrho f)(x)-f(x)]\|_E \leqq 6\sqrt{2}\,\omega_2(\varrho^{-2},f;X)_\eta.$$

In particular, if $\omega_2(t,f;X)_\eta=O(t^\alpha)$ for each $\eta>0$, where $0<\alpha\leqq 2$, i.e. $f\in \mathrm{Lip}_2(\alpha;X)$, then

$$\|e^{-\beta x^2}[(W_\varrho f)(x)-f(x)]\|_E = O(\varrho^{-\alpha}) \qquad (\forall \beta>0).$$

The latter result, a direct approximation theorem, as well as its converse, is already to be found in [10].

A more interesting related application would be the approximation of an operator $T: L^p(\mathbf{R}) \to L^q(\mathbf{R})$ by bounded operators $T_\varrho: L^p(\mathbf{R}) \to L^q(\mathbf{R})$, $\varrho>0$, by some modulus of continuity, where both T and T_ϱ are singular integrals of Fourier convolution type, i.e.

$$(Tf)(x) = \int_{-\infty}^{\infty} f(x-u)\,\chi(u)\,du, \quad (T_\varrho f)(x) = \int_{-\infty}^{\infty} f(x-u)\,\chi_\varrho(u)\,du,$$

with $\chi, \chi_\varrho \in L^r(\mathbf{R})$, $p^{-1}+r^{-1} \geqq 1$ and $q^{-1} = p^{-1}+r^{-1}-1$. An open problem here would be to express conditions (6, i, ii) or (9, i, ii) of Thm. 1 or 2 equivalently in terms of the kernels χ, χ_ϱ themselves. Whereas condition (i), namely $\chi-\chi_\varrho$ being in $L^r(\mathbf{R})$ is satisfied by assumption, (ii) would be the difficult one. A solution would deliver conditions which are not only sufficient for an estimate by some modulus of continuity but also necessary.

4. Error estimates for quadrature formulae

Our general theorem enables one to deduce estimates for numerical integration formulae as was pointed out to us by Dr. H. Esser.

For $f\in C^\mu[a,b]$, the space of μ-times ($\mu=0, 1, 2, \dots$) continuously differentiable functions on $[a,b]$, let us set (compare V. I. Krylov [11])

$$(10) \qquad Qf = \int_a^b f(x)\,dx, \quad Q_n^\mu f = \sum_{i=1}^{n} A_{i,n} f(x_{i,n}) + \sum_{\nu=1}^{\mu}\sum_{i=1}^{n} B_{i,n}^\nu f^{(\nu)}(x_{i,n}^\nu),$$

with given nodes $x_{i,n}$, $x_{i,n}^\nu \in [a,b]$ and weights $A_{i,n}$, $B_{i,n}^\nu$. Then Q and Q_n^μ define linear functionals on $C^\mu[a,b]$. In order to obtain an error estimate of Qf by $Q_n^\mu f$ for large n, one assumes that the quadrature formula $Q_n^\mu f \approx Qf$ is exact for polynomials p_m of fixed degree $m(\geqq\mu)$, i.e. $Q_n^\mu p_m = Qp_m$.

To apply Thm. 1 we take $X=C^{\mu}[a,b]$ and $A=C^{m+1}[a,b]$, $m\geqq\mu$, equipped with seminorms

$$p(f)=|f|_{C^{\mu}}\equiv\sup_x|f^{(\mu)}(x)|\qquad(f\in C^{\mu})$$

and

$$\bar p(f)=|f|_{C^{m+1}}\equiv\sup_x|f^{(m+1)}(x)|\qquad(f\in C^{m+1}),$$

respectively. In the setting of this example conditions (6, i; ii) of Thm. 1 may be rewritten as

$$|Q_n^{\mu}-Q|_{[C^{\mu},\mathbf{R}^1]}\leqq Mn^{-\mu},\quad |Q_n^{\mu}-Q|_{[C^{m+1},\mathbf{R}^1]}\leqq Dn^{-m-1}$$

with ϱ the discrete n, $\varphi(n)=n^{-\mu}$, $\psi(n)=n^{-m-1}$ and

$$|Q_n^{\mu}-Q|_{[C^l,\mathbf{R}^1]}=\sup_{\substack{f\in C^l\\ f\neq 0}}\frac{|Q_n^{\mu}f-Qf|_{\mathbf{R}^1}}{|f|_{C^l}}$$

for $l=\mu$ and $l=m+1$, respectively.

Now, in case $l\geqq 1$, these quantities may be computed with the aid of the theorem of Peano asserting that

$$\text{(11)}\quad Q_n^{\mu}f-Qf=\int_a^b f^{(l)}(t)\,\chi_{n,l-1}^{\mu}(t)\,dt\qquad(f\in C^l[a,b];\ l=\mu,\ m+1,\ l\geqq 1),$$

where

$$\text{(12)}\qquad \chi_{n,l-1}^{\mu}(t)=\frac{1}{(l-1)!}(Q_n^{\mu}-Q)_x(x-t)_+^{l-1}$$

and

$$(x-t)_+^{l-1}=\begin{cases}(x-t)^{l-1}, & x\geqq t,\\ 0\quad, & x<t,\end{cases}$$

the index x in (12) meaning that the functional $Q_n^{\mu}-Q$ is applied to $(x-t)_+^{l-1}$ with respect to x. From (11) we obtain

$$|Q_n^{\mu}-Q|_{[C^l,\mathbf{R}^1]}=\int_a^b|\chi_{n,l-1}^{\mu}(t)|\,dt\qquad(l=\mu,m+1,l\geqq 1).$$

In case $l=0$, i.e. $\mu=0$, there holds

$$|Q_n^0-Q|_{[C,\mathbf{R}^1]}=|Q_n^0|_{[C,\mathbf{R}^1]}+|Q|_{[C,\mathbf{R}^1]}=(b-a)+\sum_{i=1}^{n}|A_{i,n}|.$$

Concerning (5) of Thm. 1 we may estimate the K-functional

$$K(t^{m+1-\mu},f;\,C^{\mu}[a,b],\,C^{m+1}[a,b])$$

by the $(m+1-\mu)$-th modulus of continuity (cf. H. JOHNEN [9]):

$$\omega_{m+1-\mu}(t;f^{(\mu)}) \leqq m_1 K(t^{m+1-\mu}, f;\ C^\mu[a,b],\ C^{m+1}[a,b]) \leqq$$
$$\leqq m_2\omega_{m+1-\mu}(t;f^{(\mu)}),$$

where $(m+1-\mu=r)$

$$\omega_r(t;f^{(\mu)}) = \sup_{|s|\leqq t}\left\{\sup_{x,x+rs\in[a,b]}\left|\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f^{(\mu)}(x+ks)\right|\right\}.$$

Combining these results one obtains

Theorem 3. *Under the above definitions*

$$|Q_n^\mu f - \int_a^b f(x)\,dx| \leqq C_{\mu,m}\, n^{-\mu}\,\omega_{m+1-\mu}(n^{-1};f^{(\mu)}) \qquad (\forall f\in C^\mu[a,b])$$

holds if and only if

$$\text{(i)}\quad \left.\begin{array}{l}(\mu\geqq 1): \int_a^b |\chi_{n,\mu-1}^\mu(t)|\,dt\\ (\mu=0): \sum_{i=1}^{n}|A_{i,n}|\end{array}\right\} = O(n^{-\mu}), \qquad \text{(ii)}\quad \int_a^b |\chi_{n,m}^\mu(t)|\,dt = O(n^{-m-1}).$$

Let us note that (i) and (ii) may be verified for many examples, for instance in case $\mu=0$ for the composite Newton—Cotes formulae; cf. P. J. DAVIS—P. RABINOWITZ [6].

For such examples our result would yield error estimates for the quadrature formula $Q_n^0 f \approx Qf$ which are entirely free of derivatives. The determination of the best possible constants $C_{0,m}$ is another problem.

Derivative-free error estimates, at least in the case of functions which are analytic, were originally investigated by G. HÄMMERLIN [8a, b]. Thm. 3 may be interpreted as a result in ESSER [7] now equipped with rate. See also [7] for literature on the subject.

5. Mean ergodic theorem

This application gives part of the results obtained by P. L. BUTZER, D. LEVIATAN and U. WESTPHAL in [4, 5, 12], where the mean ergodic theorem was studied with respect to the rate of its convergence.

Let $\sigma_n^\alpha(T)$ be the Cesàro-means of order $\alpha\geqq 1$ of the iterates of a bounded linear operator T from a Banach space X (norm $\|\cdot\|$) into itself, i.e.

$$\sigma_n^\alpha(T) = \binom{n+\alpha}{n}^{-1}\sum_{i=0}^{n}\binom{n-i+\alpha-1}{n-i}T^i \quad (\alpha\geqq 1,\ n=0,1,2,\dots).$$

If $\|T^n\|_{[X,X]} \leqq M_0$, $n=0, 1, 2, \ldots$, then the mean ergodic theorem asserts

$$\lim_{n\to\infty} \|\sigma_n^\alpha(T)f - Pf\| = 0 \qquad (\forall f \in X_0),$$

where $X_0 = N(I-T) \oplus \overline{R(I-T)}$, $N(I-T)$ denoting the null space and $\overline{R(I-T)}$ the closure of the range of $(I-T)$, and P is the projection of X_0 on $N(I-T)$ parallel to $\overline{R(I-T)}$. If $T_0 = T/X_0$, define a linear operator B with domain $D(B) = N(I-T) \oplus \oplus R(I-T_0)$ and range in X_0 by $Bf=g$, where $g \in X_0$ is uniquely determined by $(I-P)f = (I-T_0)g$ and $Pg=0$.

We may then apply Thm. 1 to X_0 normed by $p(f) = \|f\|$ and $D(B)$ with semi-norm $\bar{p}(f) = \|Bf\|$. Indeed, since the following inequalities are valid (compare [5, 12])

(i) $$\|\sigma_n^\alpha(T)f - Pf\| \leqq M_0(M_0+1)\|f\| \qquad (\forall f \in X_0),$$

(ii) $$\|\sigma_n^\alpha(T)f - Pf\| \leqq \frac{\alpha}{n+1}(M_0+1)\|Bf\| \qquad (\forall f \in D(B)),$$

one concludes that

$$\|\sigma_n^\alpha(T)f - Pf\| \leqq CK(n^{-1}, f; X_0, D(B)) \qquad (\forall f \in X_0).$$

Defining a generalized Lipschitz class by

$$\mathrm{Lip}\,(\delta; X_0) = \{f \in X_0;\, K(t, f; X_0, D(B)) = O(t^\delta)\},$$

one has

Proposition 2. *If* $f \in \mathrm{Lip}\,(\delta; X_0)$, $0 < \delta \leqq 1$, *then*

$$\|\sigma_n^\alpha(T)f - Pf\| = O(n^{-\delta}).$$

References

[1] N. BOURBAKI, *Eléments de Mathématique*, Livre V: *Espaces vectoriels topologiques*, Chap. III: *Espaces d'applications linéaires continues*, Hermann (Paris, 1955).

[2] P. L. BUTZER—H. BERENS, *Semi-Groups of Operators and Approximation*, Springer (Berlin—Heidelberg—New York, 1967).

[3] P. L. BUTZER—K. SCHERER, Approximation theorems for sequences of commutative operators in Banach spaces, *Proc. Conf. Constructive Function Theory*, Varna, Bulgaria, 1970. Ed. B. Penkov—D. Vačov (Sofia, 1972), 137—145.

[4] P. L. BUTZER—U. WESTPHAL, The mean ergodic theorem and saturation, *Indiana Univ. Math. J.*, **20** (1971), 1163—1174.

[5] P. L. BUTZER—U. WESTPHAL, Ein Operatorenkalkül für das approximationstheoretische Verhalten des Ergodensatzes im Mittel, *Proc. Conf. Linear Operators and Approximation*, Oberwolfach, 1971. Ed. P. L. Butzer—B. Sz.-Nagy—J.-P. Kahane, ISNM Vol. 20, Birkhäuser (Basel, 1972).

[6] P. J. DAVIS—P. RABINOWITZ, *Numerical Integration*, Blaisdell (Waltham, Mass., 1967).

[7] K.-H. ESSER, *Neue Konvergenzsätze und Konstruktionsprinzipien zur Aufstellung von Quadraturverfahren* (Dissertation, RWTH Aachen, 1970).

[8] H. G. GARNIR—M. DE WILDE—J. SCHMETS, *Analyse Fonctionnelle*. Tome I: *Théorie générale*, Birkhäuser (Basel 1968).

[8a] G. HAMMERLIN, Über ableitungsfreie Schranken für Quadraturfehler, *Num. Math.*, **5** (1963), 225—233.

[8b] G. HÄMMERLIN, Über ableitungsfreie Schranken für Quadraturfehler. II. Ergänzungen und Möglichkeiten zur Verbesserung, *Num. Math.*, **7** (1965), 232—237.

[9] H. JOHNEN, Inequalities connected with moduli of smoothness, *Mat. Vesnik*, **9 (24)** (1972), 289—303.

[10] J. KEMPER—R. J. NESSEL, Gewichtete Approximation durch variationsvermindernde Operatoren vom Faltungstyp, *Forschungsberichte des Landes Nordrhein-Westfalen* (in print).

[11] V. I. KRYLOV, *Approximate Calculation of Integrals*, Macmillan (New York, 1962).

[12] D. LEVIATAN—U. WESTPHAL, On the mean ergodic theorem and approximation, *Mathematica* (Cluj) (in print).

[13] P. O. RUNCK, Über Konvergenzgeschwindigkeit linearer Operatoren in Banach-Räumen, *Proc. Conf. Approximation Theory*, Oberwolfach, 1963. Ed. P. L. Butzer—J. Korevaar, ISNM Vol 5, Birkhäuser (Basel, 1964).

[14] H. H. SCHAEFER, *Topological Vector Spaces*, Macmillan (New York, 1966).

[15] M. DE WILDE, Ultrabornological spaces and the closed graph theorem, *Bull. Roy. Sci. Liège*, **40** (1971), 116—118.

LEHRSTUHL A FÜR MATHEMATIK
TECHNOLOGICAL UNIVERSITY
AACHEN, GERMANY

(Received October 6, 1972)