Inner dilations of analytic matrix functions and Darlington synthesis*)

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Dedicated to Professor Béla Sz.-Nagy on his 60th birthday

0. Introduction. In this article we prove the following theorem motivated by the problem of Darlington synthesis in electrical network theory.

Theorem. *Let $S(z)$ be an analytic contractive operator-valued function on the unit disk $\boldsymbol{D}$. If $S(z)$ has a meromorphic pseudo-continuation of bounded type to the exterior $\boldsymbol{D}_e$ of the unit disk (including the point at infinity), then $S(z)$ has an inner dilation $U(z)$, that is, there exists an analytic operator-valued function $U(z)$ which is unitary-valued on the unit circle $\boldsymbol{T}$ such that*

$$U(z)=\begin{pmatrix} S(z) & A(z) \\ B(z) & C(z)\end{pmatrix}. \tag{1}$$

If $S(z)$ and $U(z)$ are both matrix-valued functions, or if $U(z)$ is required to have a scalar multiple, then this condition is necessary as well as sufficient.

If $S(z)$ is matrix-valued and has rational functions as entries, then an ordinary meromorphic continuation exists and hence a dilation is always possible in this case. The construction of such an inner function is actually described in most books on network design under the heading of Darlington or Belevitch synthesis (cf. [6]). (The results there are stated for functions on the right half plane since this is the context in which the engineers work.) Our theorem extends this classical result and generalizes a recent result of the electrical engineer DEWILDE ([1], Thm. 4).

Analytic complex-valued functions defined on $\boldsymbol{D}$ and lying in the Hardy space H^2 which possess meromorphic pseudo-continuations of bounded type to $\boldsymbol{D}_e$ have been studied in [3], where it is shown that they are precisely the non-cyclic vectors

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for the backward shift operator. Several alternate characterizations of this class of functions are given there.

The study of operator functions on $\boldsymbol{D}$ which possess meromorphic pseudo-continuations to $\boldsymbol{D}_e$ would seem to merit further study. We show in passing that inner functions have such a continuation of bounded type if and only if they have a scalar multiple.

1. Meromorphic pseudo-continuation. If $\mathfrak{H}$ is a separable complex Hilbert space, then $\boldsymbol{L}^2_{\mathfrak{H}}$ denotes the Hilbert space of norm square-integrable Lebesgue-measurable $\mathfrak{H}$-valued functions on $\boldsymbol{T}$ (see [9]). We let $\boldsymbol{H}^2_{\mathfrak{H}}$ denote the closed subspace of functions in $\boldsymbol{L}^2_{\mathfrak{H}}$ with zero Fourier coefficients of negative indices. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are separable complex Hilbert spaces, then $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ denotes the Banach space of bounded linear transformations from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. We abbreviate $\mathfrak{L}(\mathfrak{H}, \mathfrak{H})$ as $\mathfrak{L}(\mathfrak{H})$. Moreover, we let $\boldsymbol{L}^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}$ denote the Banach space of essentially bounded weakly-measurable $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$-valued functions on $\boldsymbol{T}$, while $\boldsymbol{H}^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}$ denotes the subspace of functions with zero Fourier coefficients of negative indices. We let H^∞ denote $H^\infty_{\mathfrak{L}(\mathbf{C},\mathbf{C})}$.

Functions in the Hardy spaces $\boldsymbol{H}^2_{\mathfrak{H}}$ and $\boldsymbol{H}^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}$ can be identified with the boundary values of holomorphic functions on $\boldsymbol{D}$.

We consider only those vector-valued meromorphic functions which are of "bounded type". In particular, we allow only those meromorphic functions on the exterior $\boldsymbol{D}_e$ (including the point at infinity) of $\boldsymbol{D}$ which can be expressed in the form $\frac{f}{\varphi}$ or $\frac{F}{\varphi}$, where f and F lie in $H^2_{\mathfrak{H}}(\boldsymbol{D}_e)$ and $H^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}(\boldsymbol{D}_e)$, respectively, while φ lies in $H^\infty(\boldsymbol{D}_e)$. Such functions possess radial limits a.e. and the following lemma is an easy exercise.

Lemma 1. *A function F in $\boldsymbol{L}^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}$ is the boundary values of a meromorphic function on $\boldsymbol{D}_e$ of bounded type if and only if there exists an inner (unimodular) function φ in H^∞ such that the function G defined by $G(e^{i\theta}) = \varphi(e^{i\theta})F(e^{i\theta})^*$ lies in $\boldsymbol{H}^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}$.*

An $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$-valued meromorphic function G of bounded type of $\boldsymbol{D}_e$ is said to be a *pseudo-continuation* of the holomorphic function F in $\boldsymbol{H}^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}(\boldsymbol{D})$ if

$$\lim_{r \to 1^-} F(re^{i\theta}) = \lim_{r \to 1^+} G(re^{i\theta}) \quad \text{a.e.}$$

One can show that such a function, if it exists, is unique. This generalization of the notion of continuation was introduced for scalar functions by Shapiro [8], and additional information can be found in [3].

A function Q in $\boldsymbol{H}^\infty_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}$ is said to be *outer* if the operator M_Q has dense range, where M_Q is the multiplication operator from $\boldsymbol{H}^2_{\mathfrak{H}_1}$ to $\boldsymbol{H}^2_{\mathfrak{H}_2}$ defined by

$$(M_Q f)(e^{i\theta}) = Q(e^{i\theta}) f(e^{i\theta}) \quad \text{for } f \text{ in } \boldsymbol{H}^2_{\mathfrak{H}_1}.$$

A function U in $H^{\infty}_{\mathfrak{L}(\mathfrak{H})}$ is said to be *inner* if $U(e^{i\theta})$ is a unitary operator on $\mathfrak{H}$ a.e.

Every meromorphic operator-valued function Φ on $\boldsymbol{D}_e$ can be written as the quotient of an analytic operator-valued function Θ by an analytic scalar-valued function φ. One may take φ to have its zeros (of appropriate order) at the poles of Φ. Thus, the question of whether φ can be taken to be bounded depends on the number of poles of Φ. One could probably further generalize the notion of continuation to allow an even weaker relation between the inside and outside functions when radial limits do not exist.

Although we shall make no direct use of the following observation in this note, we want to point out a relation between meromorphic pseudo-continuations and having a scalar multiple. Recall that a contractive function Θ in $H^{\infty}_{\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)}$ is said to have a *scalar multiple* if there exist contractive functions δ in H^{∞} and Ω in $H^{\infty}_{\mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)}$ such that

$$\Omega(z)\,\Theta(z) = \delta(z)\,I_{\mathfrak{H}_1} \quad \text{and} \quad \Theta(z)\,\Omega(z) = \delta(z)\,I_{\mathfrak{H}_2} \quad \text{for } z \text{ in } \boldsymbol{D}.$$

If Θ is an inner function, then we obtain $\Theta(e^{i\theta}) = \left(\Theta(e^{i\theta})^{-1}\right)^* = \left(\dfrac{\Omega(e^{i\theta})}{\delta(e^{i\theta})}\right)^*$, hence $\dfrac{\Omega(1/\bar{z})^*}{\overline{\delta(1/\bar{z})}}$ is a meromorphic pseudo-continuation of Θ to $\boldsymbol{D}_e$, of bounded type. This and the converse we state as

Proposition 2. *If Θ is an inner function in $H^{\infty}_{\mathfrak{L}(\mathfrak{H})}$, then Θ has a scalar multiple if and only if Θ possesses a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$.*

Proof. From Lemma 1 it follows that if Θ possesses such a pseudo-continuation, then there exists an inner function δ such that the function Ω defined by $\Omega(z) = \delta(z)\Theta(z)^*$ lies in $H^{\infty}_{\mathfrak{L}(\mathfrak{H})}$. Thus, Θ has a scalar multiple.

We now appeal to a result from [9, Cor V. 6. 3] or to the Schwarz reflection principle for matrix valued functions to obtain

Proposition 3. *If U is an inner function in $H^{\infty}_{\mathfrak{L}(C^n)}$, then U possesses a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$.*

Combining this with a result [9, VI. 5. 1] of Sz.-Nagy and Foiaş shows that a contraction operator lies in class C_0 if and only if its characteristic operator function is inner and has a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$. An interesting observation of Lax and Ralston is that the scattering matrix in the Lax—Phillips theory [5] has all the above properties except that the continuation is not of bounded type.

We now go on to the main theorem and refer the reader to [4] and [9] for further facts about Hardy spaces and inner and outer functions.

2. Inner dilations and the main theorem. The necessity half of the theorem now follows easily. If $S(z)$ has an inner dilation $U(z)$ with scalar multiple, then it follows from Proposition 2 that $U(z)$ has a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$ and consequently that $S(z)=PU(z)P$ does.

The sufficiency is less immediate and we begin with some definitions. If R is a contraction-valued function in $H^\infty_{\mathfrak{L}(\mathfrak{H})}$, then an isometric enlargement of R is an isometry-valued function W in $H^\infty_{\mathfrak{L}(\mathfrak{H},\mathfrak{K})}$, where $\mathfrak{K}$ is a superspace of $\mathfrak{H}$, such that $R(e^{i\theta})=QW(e^{i\theta})$ a.e., where Q is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Some contraction-valued functions have isometric enlargements and some do not. This notion is closely related to that of factorization. A non-negative operator-valued function M in $L^\infty_{\mathfrak{L}(\mathfrak{H})}$ is said to be factorable if there exists F in $H^\infty_{\mathfrak{L}(\mathfrak{H},\mathfrak{G})}$ such that $M(e^{i\theta})=F(e^{i\theta})^*F(e^{i\theta})$ a.e. The precise relation between isometric enlargement and factorization is contained in

Proposition 4. *A contraction-valued function R in $H^\infty_{\mathfrak{L}(\mathfrak{H})}$ has an isometric enlargement if and only if the function M defined by $M(e^{i\theta})=I-R(e^{i\theta})^*R(e^{i\theta})$ is factorable.*

Proof. If R has an isometric enlargement W in $H^\infty_{\mathfrak{L}(\mathfrak{H},\mathfrak{K})}$, then W can be expressed as

$$W(e^{i\theta})=\begin{pmatrix}R(e^{i\theta})\\ F(e^{i\theta})\end{pmatrix}$$

where F is in $H^\infty_{\mathfrak{L}(\mathfrak{H},\mathfrak{G})}$ and $\mathfrak{G}=\mathfrak{K}\ominus\mathfrak{H}$. Computing, we have

$$I_{\mathfrak{K}}=W(e^{i\theta})^*W(e^{i\theta})=R(e^{i\theta})^*R(e^{i\theta})+F(e^{i\theta})^*F(e^{i\theta})\quad\text{a.e.}$$

and hence the non-negative operator-valued function $M(e^{i\theta})=I-R(e^{i\theta})^*R(e^{i\theta})$ is factorable. Conversely, if M is factorable, then W can be defined in this manner.

Not every non-negative operator function is factorable and various criteria are known (cf. [2]). We add a further

Proposition 5. *If the non-negative and contractive operator-valued function M in $L^\infty_{\mathfrak{L}(\mathfrak{H})}$ is the boundary value of a meromorphic $\mathfrak{L}(\mathfrak{H})$ valued function of bounded type on $\boldsymbol{D}_e$, then M is factorable as $M(e^{i\theta})=A(e^{i\theta})^*A(e^{i\theta})$ for some outer function A which has a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$.*

Proof. Since $M\geqq M^2$ one can derive from a standard comparison theorem [2] or ([7], Theorem 1.18) that it suffices to show that M^2 is factorable. By definition there exists a scalar inner function φ such that φM lies in $H^\infty_{\mathfrak{L}(\mathfrak{H})}$. Hence, we have

$$\bigcap_{n\geqq 0}\operatorname{clos}[e^{in\theta}M(e^{i\theta})H^2_{\mathfrak{H}}]=\bar{\varphi}\{\bigcap_{n\geqq 0}\operatorname{clos}[e^{in\theta}\varphi(e^{i\theta})M(e^{i\theta})H^2_{\mathfrak{H}}]\}\subset$$

$$\subset\bar{\varphi}\{\bigcap_{n\geqq 0}\operatorname{clos}[e^{in\theta}H^2_{\mathfrak{H}}]\}=\{0\}$$

which shows that M^2 is factorable by ([9], Prop. V. 4. 2).

It remains to show that A has a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$. We have $\varphi M = \varphi A^* A$ and since φM lies in $H^\infty_{\mathfrak{L}(\mathfrak{H})}$, the operator φF^* maps $FH^2_{\mathfrak{H}}$ into $H^2_{\mathfrak{H}}$. Since F is outer, $FH^2_{\mathfrak{H}}$ is dense in $H^2_{\mathfrak{G}}$ and therefore φF^* maps $H^2_{\mathfrak{G}}$ into $H^2_{\mathfrak{H}}$. Thus φF^* lies in $H^\infty_{\mathfrak{L}(\mathfrak{G},\mathfrak{H})}$ which completes the proof.

Now we complete the proof of the main theorem. If S is a contraction-valued operator function in $H^\infty_{\mathfrak{L}(\mathfrak{H})}$ with a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$, then by Proposition 5, $I - S(e^{i\theta})^* S(e^{i\theta})$ has a factorization which is pseudo-continuable to a meromorphic operator-valued function of bounded type on $\boldsymbol{D}_e$. Thus, by Proposition 4, S has an isometric enlargement W in $H^\infty_{\mathfrak{L}(\mathfrak{H},\mathfrak{K})}$ which by construction has a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$.

If we now consider the function R in $H^\infty_{\mathfrak{L}(\mathfrak{H},\mathfrak{K})}$ defined by $R(e^{i\theta}) = W(e^{-i\theta})^*$, then R is contraction-valued and has the obvious meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$ defined in terms of that for W.

Thus, again applying Propositions 4 and 5 to R, we obtain an isometric enlargement of R to a function V in $H^\infty_{\mathfrak{L}(\mathfrak{K},\mathfrak{K}')}$. An easy argument shows that $V(e^{i\theta})$ is an isometrical isomorphism of $\mathfrak{K}$ onto $\mathfrak{K}'$ a.e. If we define the function U in $H^\infty_{\mathfrak{L}(\mathfrak{K}',\mathfrak{K})}$ such that $U(e^{i\theta}) = V(e^{-i\theta})^*$, then

$$S(e^{i\theta}) = P_{\mathfrak{H}} U(e^{i\theta})_{|\mathfrak{H}} \quad \text{a.e.},$$

$U(e^{i\theta})$ is an isometrical isomorphism of $\mathfrak{K}'$ onto $\mathfrak{K}$ a.e. and U has a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$.

If $\mathfrak{H}$ is finite-dimensional, then both $\mathfrak{K}$ and $\mathfrak{K}'$ are finite-dimensional and have the same dimension. Thus, $\mathfrak{K} \ominus \mathfrak{H}$ and $\mathfrak{K}' \ominus \mathfrak{H}$ can be identified and U is seen to give the promised unitary-valued matrix function dilating S.

If $\mathfrak{H}$ is infinite-dimensional, then it may not be true that $\mathfrak{K} \ominus \mathfrak{H}$ and $\mathfrak{K}' \ominus \mathfrak{H}$ have the same dimension. If one wants to secure a dilation of the promised form, then taking $U \oplus I_{\mathfrak{G}}$, where $\mathfrak{G}$ is an infinite dimensional Hilbert space, enables us to identify $\mathfrak{K} \oplus \mathfrak{G}$ with $\mathfrak{K}' \oplus \mathfrak{G}$ such that $\mathfrak{H}$ is identified with $\mathfrak{H}$. Thus we obtain a unitary-valued dilation for S of the desired form.

We conclude this section with a remark. If the Hilbert space $\mathfrak{H}$ is equipped with a conjugation, denoted by —, then S in $H^\infty_{\mathfrak{L}(\mathfrak{H})}$ is said to be a *real function* if $\bar{S}(e^{i\theta}) = S(e^{-i\theta})$ a.e. It is reasonable to ask that an inner dilation, if one exists, also be real. One can check that this is indeed the case by verifying that each of the operators constructed in the preceding argument is a real operator. In particular, since outer factors are unique, the outer factorization of a real non-negative operator-valued function will consist of real operators.

3. Classical inner dilations. In this section we discuss a more classical approach to obtaining an inner dilation for a matrix-valued analytic function which is closer in spirit to that used by the electrical engineers.

Given S in $H^\infty_{\mathfrak{L}(C^n)}$ satisfying the hypotheses of the theorem we seek to construct a unitary inner function U of the form

$$U(z) = \begin{pmatrix} S(z) & B(z) \\ A(z) & C(z) \end{pmatrix}.$$

First, we choose an outer function A satisfying

$$A(e^{i\theta})^* A(e^{i\theta}) = I - S(e^{i\theta})^* S(e^{i\theta}).$$

Secondly, we choose B_0 such that $B_0^\sim$ defined by $B_0^\sim(e^{i\theta}) = B_0(e^{-i\theta})^*$ is outer and B_0 satisfies

$$B_0(e^{i\theta}) B_0(e^{i\theta})^* = I - S(e^{i\theta}) S(e^{i\theta})^*.$$

Proposition 4 guarantees the existence of both A and B_0. Next we define C_0 such that

$$C_0(e^{i\theta}) = -B_0(e^{i\theta}) R(e^{i\theta})^{-1} S(e^{i\theta})^* A(e^{i\theta}),$$

where $R(e^{i\theta})^{-1}$ is the standard pseudo-inverse*) of $I - S(e^{i\theta})^* S(e^{i\theta})$. One can check that the matrix function U thus defined has unitary boundary values. There is a problem, however; U is not, in general, analytic. However, with a bit of effort one can show that R has a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$. Consequently, there exists a scalar inner function φ such that φC_0 lies in $H^\infty_{\mathfrak{L}(C^n)}$. If we set $B = \varphi B_0$ and $C = \varphi C_0$, then the matrix function defined by U has the desired properties.

We conclude with a question we have been unable to answer. What functions can occur as the 1—1 entries in a matrix representation for an operator-valued inner function? If the inner function has a scalar factor, then it is necessary and sufficient that the function have a meromorphic pseudo-continuation of bounded type to $\boldsymbol{D}_e$. If we assume the function to have modulus uniformly bounded away from 1, then a modification of the preceding construction argument can be used to provide an answer.

Added in proof. We wish to thank Professor DOUGLAS N. CLARK for calling our attention to the fact that results similar to those obtained in this paper are announced in D. Z. AROV, Darlington's method for dissipative systems, *Doklady Akad. Nauk SSSR*, **201** (1971), 559—562 (Russian); *Soviet Phys. Doklady*, **16** (1971), 954—956 (1972) (English translation).

*) The standard pseudo-inverse of a matrix M is the matrix X with the property that XM is the projection onto [null M]$^\perp$ and MX is the pr ojection onto range [M].

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