On the greatest zero of an orthogonal polynomial. I

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Dedicated to Prof. Béla Szőkefalvi-Nagy on the occasion of his 60th birthday

0. Introduction

Let $w(x)$ $(-\infty<x<\infty)$ be an *even* weight function, and let $\{p_n(w;x)=\gamma_n(w)x^n+\cdots;\ n=0,1,\ldots\}$ be the sequence of orthonormal polynomials with respect to w, i.e.

(1) $$\int_{-\infty}^{\infty} p_m(w;x)p_n(w;x)\,w(x)\,dx=\begin{cases}0 & (m\neq n),\\ 1 & (m=n).\end{cases}$$

Moreover, let $X_n(w)=x_{1n}(w)$ be the greatest zero of $p_n(w;x)$. In part 1 of the present note we express the order of $X_n(w)$ with the aid of the sequence $\{\gamma_\nu(w)\}$ (see Theorem 1). After deducing some lemmas in part 2, we apply this result in part 3 to the weight

(2) $$w_{\varrho,2k}(x)=|x|^{\varrho}e^{-x^{2k}}$$

where $\varrho\geqq 0$ and k is a positive integer. We prove the estimate $\gamma_{\nu-1}(w_{\varrho,2k})/\gamma_\nu(w_{\varrho,2k})=O(\nu^{1/2k})$ which seems far from being trivial and conclude from it that

(3) $$X_n(w_{\varrho,2k})\sim n^{1/2k}\sim\gamma_{n-1}(w_{\varrho,2k})/\gamma_n(w_{\varrho,2k}).$$

The relation (3) has several interesting implications in approximation theory; we hope to return to them soon.

1. An inequality on $X_n(w)$

Theorem 1. *For every even weight function $w(x)$ we have*

(4) $$\max_{1\leqq k\leqq n-1}\frac{\gamma_{k-1}(w)}{\gamma_k(w)}\leqq X_n(w)\leqq 2\max_{1\leqq k\leqq n-1}\frac{\gamma_{k-1}(w)}{\gamma_k(w)}.$$

Remarks. a) Let $w_0(x)=(1-x^2)^{-1/2}$ with support $[-1,1]$. Then the first three orthogonal polynomials are $\frac{1}{\sqrt{\pi}},\sqrt{\frac{2}{\pi}}\,x,\sqrt{\frac{2}{\pi}}\,(2x^2-1)$, i.e. $\gamma_0(w_0)/\gamma_1(w_0)=$

$= \frac{1}{\sqrt{2}} = X_2(w_0)$. This example shows that the left-hand part of inequality. (4) is precise.

b) In case $w_{02}(x) = e^{-x^2}$ the orthogonal polynomials $p_n(w)$ are the orthonormal Hermite polynomials $h_n(x)$, so that $\gamma_{n-1}(w_{02})/\gamma_n(w_{02}) = \sqrt{\frac{n}{2}}$ and $X_n(w_{02}) \approx \sqrt{2n}$. This example shows that the factor 2 on the right-hand side of (4) can not be replaced by any smaller constant.

Proof. By a classical result of P. L. CHEBYCHEV (see G. SZEGŐ [2], 7. 7. 2) we have

$$X_n(w) = \max \frac{\int_{-\infty}^{\infty} x [P_{n-1}(x)]^2 w(x)\, dx}{\int_{-\infty}^{\infty} [P_{n-1}(x)]^2 w(x)\, dx}, \tag{5}$$

where $P_{n-1}(x)$ runs over all polynomials of degree $\leqq n-1$. Let us represent $P_{n-1}(x)$ as

$$P_{n-1}(x) = \sum_{j=0}^{n-1} c_j p_j(w; x). \tag{6}$$

We recall that by the recursion formula applied to even w we have

$$x p_j(w; x) = \frac{\gamma_j(w)}{\gamma_{j+1}(w)} p_{j+1}(w; x) + \frac{\gamma_{j-1}(w)}{\gamma_j(w)} p_{j-1}(w; x) \tag{7}$$

(see e.g. G. FREUD [1], § I. 2).

Inserting (6) into (5) and taking (1) and (7) into consideration we obtain

$$X_n(w) = 2 \max \frac{\sum_{k=1}^{n-1} \frac{\gamma_{k-1}(w)}{\gamma_k(w)} c_{k-1} c_k}{\sum_{k=0}^{n-1} c_k^2}, \tag{8}$$

where all the c_k $(k = 0, 1, \dots, n-1)$ run idependently over the reals. Inserting $c_{j-1} = c_j = 1$ and $c_k = 0$ if $k \neq j-1, j$ into the expression on the right of (8), we obtain

$$X_n(w) \geqq \frac{\gamma_{j-1}(w)}{\gamma_j(w)} \qquad (j = 1, 2, \dots n-1). \tag{9}$$

In turn, by Cauchy's inequality for every $\{c_k\}$ we have

$$\sum_{k=1}^{n-1} \frac{\gamma_{k-1}(w)}{\gamma_k(w)} c_{k-1} c_k \leqq \max_{1 \leqq k \leqq n-1} \frac{\gamma_{k-1}(w)}{\gamma_k(w)} \sum_{k=0}^{n-1} |c_{k-1} c_k| \leqq$$

$$\leqq \max_{1 \leqq k \leqq n-1} \frac{\gamma_{k-1}(w)}{\gamma_k(w)} \sum_{k=1}^{n-1} c_k^2.$$

The left-hand side of (4) is implied by (9) and the right hand side of (4) is a consequence of (8) and (10), and so Theorem 1 is proved.

2. Lemmata

Let

(11) $$w_{\varrho\beta}(x) = |x|^{\varrho} e^{-|x|^{\beta}} \qquad (-\infty < x < \infty).$$

Lemma 1. *For every $\varrho \geqq 0$ and $\beta > 0$ we have*

(12) $$\frac{\gamma_n(w_{\varrho\beta})}{\gamma_{n-1}(w_{\varrho\beta})} = \frac{\beta}{n + \Delta_n \varrho} \int_{-\infty}^{\infty} p_n(w_{\varrho\beta}; x) p_{n-1}(w_{\varrho\beta}; x) x^{-1} |x|^{\beta} w_{\varrho\beta}(x)\, dx,$$

where

(13) $$\Delta_n = \tfrac{1}{2}[1 + (-1)^{n+1}].$$

Proof. We have

$$\int_{-\infty}^{\infty} p_n'(w_{\varrho\beta}; x) p_{n-1}(w_{\varrho\beta}; x) w_{\varrho\beta}(x)\, dx =$$

$$= \int_{-\infty}^{\infty} [n\gamma_n(w_{\varrho\beta}) x^{n-1} + \cdots] p_{n-1}(w_{\varrho\beta}; x) w_{\varrho\beta}(x)\, dx =$$

$$= \int_{-\infty}^{\infty} \left[n \frac{\gamma_n(w_{\varrho\beta})}{\gamma_{n-1}(w_{\varrho\beta})} p_{n-1}(w_{\varrho\beta}; x) + P_{n-2}(x) \right] p_{n-1}(w_{\varrho\beta}; x) w_{\varrho\beta}(x)\, dx,$$

where P_{n-2} is a polynomial of degree $\leqq n-2$. Applying the orthogonality relations (1), we get

(14) $$n \frac{\gamma_n(w_{\varrho\beta})}{\gamma_{n-1}(w_{\varrho\beta})} = \int_{-\infty}^{\infty} p_n'(w_{\varrho\beta}; x) p_{n-1}(w_{\varrho\beta}; x) w_{\varrho\beta}(x)\, dx.$$

Partial integration gives

$$\int_{-\infty}^{\infty} p_n'(w_{\varrho\beta};x)p_{n-1}(w_{\varrho\beta};x)w_{\varrho\beta}(x)\,dx = -\int_{-\infty}^{\infty} p_n(w_{\varrho\beta};x)[p_{n-1}(w_{\varrho\beta};x)w_{\varrho\beta}(x)]'\,dx =$$

$$= \beta\int_{-\infty}^{\infty} p_n(w_{\varrho\beta};x)p_{n-1}(w_{\varrho\beta};x)x^{-1}|x|^{\beta}w_{\varrho\beta}(x)\,dx - \tag{15}$$

$$-\varrho\int_{-\infty}^{\infty} p_n(w_{\varrho\beta};x)p_{n-1}(w_{\varrho\beta};x)x^{-1}w_{\varrho\beta}(x)\,dx,$$

since $\int p_n(w)p_n'(w)w\,dx=0$ by (1).

If n is even, $p_{n-1}(w_{\varrho\beta};x)$ is odd, and so $x^{-1}p_{n-1}(w_{\varrho\beta};x)$ is a polynomial of degree $n-2$. Consequently, the second integral on the right of (15) vanishes by (1). In this way, from (15) we obtained

$$\begin{aligned}&\int_{-\infty}^{\infty} p_n'(w_{\varrho\beta};x)p_{n-1}(w_{\varrho\beta};x)w_{\varrho\beta}(x)\,dx = \\ &= \beta\int_{-\infty}^{\infty} p_n(w_{\varrho\beta};x)p_{n-1}(w_{\varrho\beta};x)x^{-1}|x|^{\beta}w_{\varrho\beta}(x)\,dx. \qquad (n \text{ is even}).\end{aligned} \tag{16}$$

Let now n be odd. Then $p_n(w_{\varrho\beta};x)$ is odd, and so

$$x^{-1}p_n(w_{\varrho\beta};x) = \frac{\gamma_n(w_{\varrho\beta})}{\gamma_{n-1}(w_{\varrho\beta})}p_{n-1}(w_{\varrho\beta};x)+P_{n-2}(x),$$

where $P_{n-2}(x)$ is a polynomial of degree $\leqq n-2$. Using the orthogonality relation (1) we see that

$$\int_{-\infty}^{\infty} p_n(w_{\varrho\beta};x)p_{n-1}(w_{\varrho\beta};x)x^{-1}w_{\varrho\beta}(x)\,dx = \frac{\gamma_n(w_{\varrho\beta})}{\gamma_{n-1}(w_{\varrho\beta})}. \qquad (n \text{ is odd}) \tag{17}$$

From (14), (15), (16), and (17) we see that (12) holds for both even and odd integers n. Q.E.D.

Lemma 2. *For every positive integer k we have*

$$\left[\frac{\gamma_{n-1}(w_{\varrho,2k})}{\gamma_n(w_{\varrho,2k})}\right]^{2k} \leqq \frac{n+\varrho\Delta_n}{2k}. \tag{18}$$

Remark. For $k=1$, $\varrho=0$ we have equality in (18).

Proof. We infer by induction from the recursion formula (7) that, for every positive integer l and every *even* w, we have

(19) $$x^l p_n(w;x)=\sum_{j=0}^{n+l} A_{n,l,j}(w)p_j(w;x),$$

where *all coefficients* $A_{n,l,j}(w)$ *are nonnegative.*

By (7) we have

(20) $$A_{n,1,n-1}(w)=\frac{\gamma_{n-1}(w)}{\gamma_n(w)}.$$

Moreover, by a repeated application of the recursion formula (7) we obtain

(21) $$A_{n-1,2,n-1}(w)=\int_{-\infty}^{\infty} x^2 p_{n-1}^2(w;x)\,w(x)\,dx=\frac{\gamma_{n-1}^2(w)}{\gamma_n^2(w)}+\frac{\gamma_{n-2}^2(w)}{\gamma_{n-1}^2(w)}\geqq\frac{\gamma_{n-1}^2(w)}{\gamma_n^2(w)}.$$

Multiplying (19) by x^2 and then applying the special case $l=2$ of the same formula to the right-hand side, we get

(22) $$A_{n,l+2,n-1}(w)\geqq A_{n,l,n-1}(w)A_{n-1,2,n-1}(w).$$

From (21) and (22) we infer by induction that

(23) $$\int_{-\infty}^{\infty} x^{2s-1}p_n(w;x)p_{n-1}(w;x)\,w(x)\,dx=A_{n,2s-1,n-1}(w)\geqq \geqq\left[\frac{\gamma_{n-1}(w)}{\gamma_n(w)}\right]^{2s-1}\qquad (s=1,2,\ldots).$$

Let us now insert $\beta=2k$ in (12) and $w=w_{\varrho,2k}$ in (23). Combining the two formulas so obtained we get (18). Q.E.D.

We introduce the moments

(24) $$\mu_r(w)=\int_{-\infty}^{\infty} x^r w(x)\,dx\qquad (r=0,1,\ldots).$$

Lemma 3. *For every even* w, *we have*

(25) $$[X_n(w)]^2\geqq \mu_{2n-2}(w)/\mu_{2n-4}(w).$$

Proof. Denoting by $X_n(w)=x_{1n}>x_{2n}>\cdots>x_{nn}=-X_n(w)$ the zeros of $p_n(w;x)$, by the Gauss—Jacobi quadrature formula we have

$$\mu_{2n-2}(w)=\sum_{j=1}^{n}\lambda_{jn}(w)x_{jn}^{2n-2}\leqq [X_n(w)]^2\sum_{j=1}^{n}\lambda_{jn}(w)x_{jn}^{2n-4}=[X_n(w)]^2\mu_{2n-4}(w).$$

Q.E.D.

We can also see that the sign of equality is valid in (25) iff $n=2$.

3. Estimates for $X_n(w_{\varrho, 2k})$

Theorem 2. *For every $\varrho \geqq 0$ and $\beta > 0$ we have*

$$\varliminf_{n\to\infty} n^{-1/\beta} X_n(w_{\varrho\beta}) \geqq (2/\beta)^{1/\beta}. \tag{26}$$

Proof. We have

$$\mu_{2r}(w_{\varrho\beta}) = 2\int_0^\infty x^{2r+\varrho} e^{-x^\beta}\, dx = \Gamma\left(\frac{2r+\varrho+1}{\beta}\right) \qquad (r = 0, 1, \ldots). \tag{27}$$

Insert (27) in (25) and apply Stirling's formula to get the desired result.

Theorem 3. *For every $\varrho \geqq 0$ and every positive integer k we have*

$$X_n(w_{\varrho, 2k}) \leqq 2(n/2k)^{1/2k}. \tag{28}$$

Remark. We have $X_n(w_{0,2}) \approx \sqrt{2n}$. So the factor on the right of (28) cannot be replaced by any constant smaller than 2.

Proof. This is a consequence of Theorem 1 and Lemma 2. We see from Theorem 2 and Theorem 3 that

$$X_n(w_{\varrho, 2k}) \sim n^{1/2k} \tag{29}$$

holds for every $\varrho \geqq 0$ and every positive integer k.

Theorem 4. *For every $\varrho \geqq 0$ and every positive integer k we have*

$$\frac{\gamma_{n-1}(w_{\varrho, 2k})}{\gamma_n(w_{\varrho, 2k})} \geqq 2^{-2k+1}\left(\frac{n}{2k}\right)^{1/2k}\left(1+\frac{k}{n}\right)^{\frac{2k-1}{2k}}. \tag{30}$$

Remark. From (30) and the combination of (28) and the left hand side of (4) we see that

$$\frac{\gamma_{n-1}(w_{\varrho, 2k})}{\gamma_n(w_{\varrho, 2k})} \sim n^{1/2k}. \tag{31}$$

Proof. Consider formula (12). The expression $p_n(w_{\varrho, 2k}; x) p_{n-1}(w_{\varrho, 2k}; x) x^{2k-1}$ is a polynomial of degree $2n+2k-2 < 2(n+k)-1$. Consequently the integral in (12) can be calculated by the Gauss—Jacobi quadrature formula over the zeros of $p_{n+k}(w_\varrho; x)$:

$$\begin{aligned} \frac{n}{2k}\frac{\gamma_n(w_{\varrho, 2k})}{\gamma_{n-1}(w_{\varrho, 2k})} &\leqq \frac{n+\varrho\Delta_n}{2k}\frac{\gamma_n(w_{\varrho, 2k})}{\gamma_{n-1}(w_{\varrho, 2k})} = \\ &= \sum_{j=1}^{n+k} \lambda_{j,n+k}(w_{\varrho, 2k})\, x_{j,n+k}^{2k-1} p_n(w_{\varrho, 2k}; x_{j,n+k}) p_{n-1}(w_{\varrho, 2k}; x_{j,n+k}) \leqq \\ &\leqq [X_{n+k}(w_{\varrho, 2k})]^{2k-1} \left\{\sum_{j=1}^{n+k} \lambda_{j,n+k}(w_{\varrho, 2k}) p_n^2(w_{\varrho, 2k}; x_{j,n+k}) \times \right. \\ &\left. \times \sum_{j=1}^{n+k} \lambda_{j,n+k}(w_{\varrho, 2k}) p_{n-1}^2(w_{\varrho, 2k}; x_{j,n+k})\right\}^{1/2} = [X_{n+k}(w_{\varrho, 2k})]^{2k-1}, \end{aligned} \tag{32}$$

since by the quadrature formula we have

$$\sum_{j=1}^{n+k} \lambda_{j,n+k}(w_{\varrho,2k})\, p_r^2(w_{\varrho,2k}; x_{j,n+k}) = \int_{-\infty}^{\infty} p_r^2(w_{\varrho,2k}; x)\, w_{\varrho,2k}(x)\, dx = 1 \qquad (r = n-1, n).$$

Inserting estimate (28) into the right-hand side of (32), we obtain the desired estimate (30) after reshuffling the factors. Q.E.D.

Literature

[1] G. FREUD, *Orthogonale Polynome*. Joint edition of Birkhäuser Verl., Deutscher Verl. der Wiss.' and Akadémiai Kiadó (Basel, Berlin, and Budapest, 1969); English translation: Pergamon Press (New York—Toronto—London, 1971).

[2] G. SZEGŐ, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ., vol. 23, 2nd ed. (New York, 1959).

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