Limits of shifts

By P. R. HALMOS in Bloomington (Indiana, U.S.A.)

Dedicated to B. Sz.-Nagy on his sixtieth birthday, July 29, 1973

What is the closure of the unilateral shifts?

The question looks odd; by long-standing tradition the unilateral shift is regarded as one operator, not a set of operators. On the occasions when the plural is used it usually indicates multiplicities, or weights, but neither of those is what is meant here. A moment's thought reveals that the question makes unambiguous sense. An operator S on a Hilbert space H is a unilateral shift (of multiplicity 1) in case there exists an orthonormal basis $\{e_0, e_1, e_2, \dots\}$ for H such that $Se_n = e_{n+1}$, $n=0, 1, 2, \dots$. From this point of view there are as many unilateral shifts of multiplicity 1 as there are orthonormal bases enumerated by the natural numbers. The problem is to determine the closure of the set of all such shifts with respect to the norm topology of operators.

The same question can be asked and the same comments can be made about bilateral shifts, which shift an orthonormal basis enumerated by all integers.

Unilateral shifts are isometric, and, therefore, so are their limits. (Reason: if $S_n \to T$, then $S_n^* S_n \to T^* T$.) If, moreover, all the terms of a convergent sequence of unilateral shifts have the same multiplicity, then the co-rank of the limit is equal to that common multiplicity. (Reason: for n large, the projections $1 - S_n S_n^*$ and $1 - TT^*$ are near, and, therefore, they have the same rank; the rank of $1 - S_n S_n^*$ is the multiplicity of S_n.)

Bilateral shifts are unitary, and, therefore, so are their limits. Since, moreover, the spectrum of every bilateral shift is the entire unit circle, it follows that the spectrum of a limit of bilateral shifts is also the entire unit circle. (Reason: the spectrum is upper semicontinuous, [6, Problem 86].)

The preceding two paragraphs describe some necessary conditions that limits of shifts must satisfy; it is natural to ask how near those conditions come to being sufficient. Can a limit of unilateral shifts of multiplicity 1, say, have a unitary direct

*) Research supported in part by a grant from the National Science Foundation.

summand? Can a limit of bilateral shifts of multiplicity 1 have anything other than an absolutely continuous spectral measure of uniform multiplicity? On first consideration both questions seem to call for a negative answer. It is remarkable, however, that the already stated necessary conditions turn out to be sufficient also. The facts are described in the following statement; the main purpose of the sequel is to prove it.

Theorem. *On a separable Hilbert space the norm closure of the set of unilateral shifts of multiplicity n $(1 \leqq n \leqq \infty)$ is the set of all isometries of co-rank n, and the norm closure of the set of bilateral shifts of multiplicity n $(1 \leqq n \leqq \infty)$ is the set of all unitary operators whose spectrum is the entire unit circle.*

The proof uses a slight sharpening of the proof of a result of R. G. Douglas (which will be described later). That result became part of the oral tradition sometime in 1971. I learned the statement from P. A. Fillmore and the proof of the central lemma (which appears as Lemma 2 below) from I. D. Berg. A treatment of the Douglas result in an extended context is to appear later [3]. The present sharpening is applied, along the way, to the proof of a theorem of von Neumann's (the so-called von Neumann converse of Weyl's theorem [11]). The result (Lemma 4) is a quantitative improvement of von Neumann's theorem for a large class of normal operators (the ones for which the spectrum coincides with the essential spectrum).

Lemma 1. *If A is a normal operator on a separable Hilbert space, then $A = D+C$, where D is diagonal, with its spectrum included in that of A, and C is compact, with its norm arbitrarily small.*

Except for the statement about the spectrum of D, this is the Berg extension [2] to normal operators of the Weyl—von Neumann theorem [11] for Hermitian ones. In my subsequent proof [9] no restriction was placed on the spectrum of D or on the size of C. There is perhaps some merit in knowing that the restrictions can be captured in the framework of that proof; the next two paragraphs show how that can be done.

As far as the size of C is concerned, the result in the Hermitian case goes back to von Neumann [11], who proved that the compact summand of a Hermitian operator could in fact be made a Hilbert—Schmidt operator with arbitrarily small Hilbert—Schmidt norm. (Cf. also [8, p. 904].) To extend the result to the normal case, use the fact that if A is normal, then $A=\varphi(A')$, where A' is Hermitian and φ is continuous [9]. Recall now that the mapping $X \mapsto \varphi(X)$, defined for each Hermitian operator X whose spectrum is in the domain of φ, is continuous in the norm topology. (This is an easy exercise whose proof uses nothing more than the Weierstrass polynomial approximation theorem and the norm continuity of the algebraic operations on operators. The statement is true for continuous functions of normal operators, as well as Hermitian ones; the only additional technique needed is the planar version of the Weierstrass theorem.) Consequence: if $A' = D' + C'$, with D' diagonal and C' compact, the norm of the (compact) operator $C = A - D (= \varphi(A') - \varphi(D'))$ can be made as small as desired by making $\|C'\|$ small enough. (Observe that because of the passage to a limit implied by the formation of a continuous function, the Hilbert—Schmidt character of the compact summand cannot automatically be asserted in the normal case. It is not known whether the reason is in the proof or in the facts.)

The problem of putting the spectrum of D into the spectrum of A can be handled as follows. For each positive number δ, there can be only finitely many eigenvalues of D farther than δ from the spectrum of A. (Reason: otherwise the eigenvalues of D would have a cluster point not in the spectrum of A, in contradiction to the fact that A and D have the same essential spectrum.) Suppose now that $A = D + C$, with D diagonal, C compact, and $\|C\|$ small enough for two purposes: (1) if the ultimate C is to have norm below ε, make the present one have norm below $\varepsilon/2$, and (2) use the upper semicontinuity of the spectrum [6, Problem 86] to guarantee that if $\|A - X\| < \|C\|$, then the spectrum of X is in the $\varepsilon/2$ neighborhood of the spectrum of A. Consider, successively, the values of δ equal to $\|C\|, \|C\|/2, \|C\|/3, \ldots$, and, each time, replace the eigenvalues of D outside the δ neighborhood of the spectrum of A by numbers in the spectrum as near as possible. The total alteration is compact and has norm not more than $\varepsilon/2$. Absorb it in C, increasing $\|C\|$ thereby to ε at worst. In case C happened to have not only small norm but small Hilbert—Schmidt norm as well, the altered C will have the same property.

Lemma 2. *If S is a shift of multiplicity 1 (unilateral or bilateral), if $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ is a sequence of complex numbers of modulus 1, and if $\varepsilon > 0$, then there exist operators D and E such that D is diagonal, with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$, and such that $S = (D \oplus E) + C$, where C is a Hilbert—Schmidt operator with Hilbert—Schmidt norm not greater than ε.*

To prove the lemma, consider an orthonormal set $\{e_0, e_1, e_2, \ldots\}$ that S shifts. If $|\lambda| = 1$, $m = 0, 1, 2, \ldots$, and $n = 1, 2, 3, \ldots$, write

$$f = (1/\sqrt{n})(e_m + e_{m+1}/\lambda + \cdots + e_{m+n-1}/\lambda^{n-1}).$$

Clearly $\|f\| = 1$. Since

$$Sf = (1/\sqrt{n})(e_{m+1} + e_{m+2}/\lambda + \cdots + e_{m+n}/\lambda^{n-1})$$

$$= \lambda(1/\sqrt{n})(e_{m+1}/\lambda + e_{m+2}/\lambda^2 + \cdots + e_{m+n}/\lambda^n),$$

it follows that

$$Sf - \lambda f = (1/\sqrt{n})(e_{m+n}/\lambda^{n-1} - \lambda e_m),$$

and hence that

$$\|Sf-\lambda f\|^2 = 2/n.$$

In other words, the vector $f=f(\lambda, m, n)$ is an approximate eigenvector for S, with approximate eigenvalue λ and degree of approximation $\sqrt{2/n}$. Since

$$S^*f-\bar{\lambda}f = -\bar{\lambda}S^*(Sf-\lambda f),$$

it follows that f is, at the same time, an approximate eigenvector for S^*, with approximate eigenvalue $\bar{\lambda}$ and degree of approximation $\sqrt{2/n}$.

The preceding construction can be applied to each of the given numbers λ_k. Choose n_k so that

$$\Sigma_k(2/n_k) \leqq (\varepsilon/2)^2,$$

and choose m_k so that the index intervals $[m_k, m_k+n_k-1]$ are pairwise disjoint. If $f_k=f(\lambda_k; m_k, n_k)$, then $\{f_1, f_2, f_3, \ldots\}$ is an orthonormal sequence such that $\Sigma_k\|Sf_k-\lambda_k f_k\|^2 \leqq (\varepsilon/2)^2$; $\Sigma_k\|S^*f_k-\bar{\lambda}_k f_k\|^2 \leqq (\varepsilon/2)^2$. Let M be the span of $\{f_1, f_2, f_3, \ldots\}$, let P be the projection with range M, and let D be the diagonal operator defined on M by

$$Df_k = \lambda_k f_k, \quad k = 1, 2, 3, \ldots .$$

Write the Hilbert space as $M\oplus M^\perp$, and, correspondingly, consider the matrices

$$S = \begin{pmatrix} X & Y \\ Z & E \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} X-D & Y \\ Z & 0 \end{pmatrix}.$$

Assertion: $X-D$, Y, and Z are Hilbert—Schmidt operators, and the sum of the squares of their Hilbert—Schmidt norms is not more than ε^2. Indeed:

$$\|(X-D)f_k\|^2+\|Zf_k\|^2 = \|(S-Q)f_k\|^2 = \|Sf_k-\lambda_k f_k\|^2 = 2/n_k,$$

and, similarly,

$$\|(X^*-D^*)f_k\|^2+\|Y^*f_k\|^2 = 2/n_k;$$

the proof of Lemma 2 is complete.

Remark. The control that the proof gives over the differences $Sf_k-\lambda_k f_k$ is strong enough to make it possible to put C into the trace class. Indeed: choose the n_k's so large that $\Sigma_k\sqrt{2/n_k}$ is small, and apply the lemma of Dunford and Schwartz [4, p. 1116] according to which $\Sigma_k\|Tf_k\|<\infty$, for an orthonormal basis $\{f_1, f_2, f_3, \ldots\}$, implies that T is in the trace class. (The statement in [4] does not seem to be formulated for the right set of exponents. In any event, it is true and the proof is valid for the exponent $p=1$.)

Lemma 3. *If $\{a_n\}$ and $\{b_n\}$ are sequences with the same cluster set C in a compact metric space, if $a_n \in C$ and $b_n \in C$ for all n, and if $\varepsilon > 0$, then there exists a permutation π of the natural numbers such that $\Sigma_n d(a_n, b_{\pi n}) < \varepsilon$.*

It is convenient to have a word to describe the sequences that occur in this statement: call a sequence recurrent if each of its terms is a cluster point of it. (The "cluster set" of a sequence is, of course, the set of all cluster points. In this language a sequence is recurrent if it is included in its own cluster set.) Lemma 3 is a sharpened version of the von Neumann permutation theorem [7], which is used in the proof of the von Neumann converse of Weyl's theorem. The original version does not assume that the given sequences are recurrent, and cannot conclude that, after the permutation, the sum of distances is small. If, for instance, $a_1 = 1$, $a_n = 0$ for $n > 1$, and $b_n = 0$ for all n, then, clearly, there is no permutation π such that $d(a_n, b_{\pi n}) \leqq 1/2$ for all n. The trouble is not that the ranges of the sequences are different; if $b_1 = b_2 = 1$ and $b_n = 0$ for $n > 2$, the inequalities $d(a_n, b_{\pi n}) \leqq 1/2$ for all n can still not be achieved. The trouble is that the cluster sets (which, to be sure, are the same) do not contain all the terms; in the first example one of the sequences fails to be recurrent, and in the second example they both do.

Now for the proof of Lemma 3.

Write $\sigma(1) = 1$. Since $a_{\sigma(1)} \in C$, there exists an index $\tau(1)$ such that $d(a_{\sigma(1)}, b_{\tau(1)}) \leqq \varepsilon/2$. Let $\tau(2)$ be the smallest index distinct from $\tau(1)$ (so that, typically, $\tau(2)$ will be 1). Since $b_{\tau(2)} \in C$, there exists an index $\sigma(2)$ distinct from $\sigma(1)$ such that $d(a_{\sigma(2)}, b_{\tau(2)}) \leqq \varepsilon/4$. The preceding four sentences describe a two-step process that is now to be applied infinitely often. The second application will indicate how the general one is to be made. Let $\sigma(3)$ be the smallest index not contained in $\{\sigma(1), \sigma(2)\}$. Find $\tau(3)$ not contained in $\{\tau(1), \tau(2)\}$ so that $d(a_{\sigma(3)}, b_{\tau(3)}) \leqq \varepsilon/8$. Let $\tau(4)$ be the smallest index not contained in $\{\tau(1), \tau(2), \tau(3)\}$. Find $\sigma(4)$ not contained in $\{\sigma(1), \sigma(2), \sigma(3)\}$ so that $d(a_{\sigma(4)}, b_{\tau(4)}) \leqq \varepsilon/16$.

When, ultimately, $\sigma(n)$ and $\tau(n)$ are defined for all n, each of σ and τ is a permutation of the set of all natural numbers. Indeed: since the definition of $\sigma(n)$ guarantees that $\sigma(n)$ is not contained in $\{\sigma(1), \ldots, \sigma(n-1)\}$, $n > 1$, the mapping σ is one-to-one; the definition for odd values of n guarantees that every natural number is in the range of σ. The argument for τ is, of course, the same, except that "odd" has to be replaced by "even".

The result is a pair of permutations σ and τ such that $d(a_{\sigma(n)}, b_{\tau(n)}) \leqq \varepsilon/2^n$ for all n. If π is defined so that $\tau(n) = \pi(\sigma(n))$ for all n, i.e., if $\pi = \tau\sigma^{-1}$, then $\Sigma_n d(a_n, b_{\pi(n)}) = \Sigma_n d(a_{\sigma(n)}, b_{\tau(n)}) \leqq \varepsilon$.

My original statement of Lemma 3 had "$d(a_n, b_{\pi(n)}) \to 0$ and $d(a_n, b_{\pi(n)}) \leqq \varepsilon$ for all n" instead of "$\Sigma_n d(a_n, b_{\pi(n)}) \leqq \varepsilon$", and my proof of it was longer; the simplification is due to J. G. Stampfli.

To apply Lemma 3, I introduce a new concept: a normal operator is *essential* if its spectrum is the same as its essential spectrum.

Lemma 4. *If A and B are essential normal operators with the same spectrum, on a separable Hilbert space, and if $\varepsilon > 0$, then there exists a unitary operator U and a compact operator K such that $A = U^* BU + K$ and $\|K\| \leqq \varepsilon$.*

Lemma 4 is a sharpened version of the von Neumann converse of Weyl's theorem. The original version does not assume that the given operators are essential, and cannot conclude that, after the unitary equivalence, they are within ε of one another. VON NEUMANN [11] remarked that, in fact, if a single compact operator is excluded from the competition, the conclusion becomes false. The point of Lemma 4 is that for essential normal operators the compact operators that appear can be made to satisfy severe and useful size restrictions.

To prove Lemma 4, use Lemma 1 to write $A = D_A + C_A$ and $B = D_B + C_B$, where D_A and D_B are diagonal, with each diagonal entry in the common spectrum, and C_A and C_B are compact, with $\|C_A\| \leqq \varepsilon/3$, $\|C_B\| \leqq \varepsilon/3$. Since D_A and A have the same essential spectrum, and since the essential spectrum of D_A is the cluster set of the diagonal, it follows that that cluster set is the common spectrum of A and B. (This step uses the assumption that A is essential.) Similarly the cluster set of D_B is that common spectrum. By Lemma 3 there exists a unitary operator U (induced by a permutation) and a compact operator C such that $D_A = U^* D_B U + C$ and $\|C\| \leqq \varepsilon/3$. Consequence:

$$A = D_A + C_A = U^* D_B U + C + C_A = U^*(B - C_B) U + C + C_A =$$
$$= U^* BU - U^* C_B U + C + C_A;$$

since $-U^* C_B U + C + C_A$ is compact and has norm not greater than ε, the proof of Lemma 4 is complete.

Remark. In case A and B are such that C_A and C_B can be made to have small Hilbert—Schmidt norm (e.g., in case A and B are Hermitian or unitary), then K can be made to have small Hilbert—Schmidt norm; the perturbation C that Lemma 3 introduces belongs, in fact, to the trace class.

The statement of Lemma 4 does not include the von Neumann converse (for not necessarily essential operators) as a special case, but the proof of Lemma 4 is, in spirit, the same as that of the unmodified version; cf. [1], [10], [11]. The main difference is that the present proof uses the quantitative version (Lemma 3) of the von Neumann permutation theorem.

For some of the statements that follow it is convenient to introduce a shorthand notation: if A and B are operators and $\varepsilon > 0$, write

$$A \sim B \quad (\varepsilon)$$

in case there exists an operator B', unitarily equivalent to B, such that $A-B'$ has Hilbert—Schmidt norm not greater than ε. (The operators A and B need not even be defined on the same Hilbert space. The generality gained thereby is shallow but useful.)

Lemma 5. *If S is a shift of multiplicity n ($1 \leqq n \leqq \infty$) (unilateral or bilateral), if U is a unitary operator on a separable Hilbert space, and if $\varepsilon > 0$, then*

$$S \sim U \oplus S \quad (\varepsilon).$$

For the proof, let $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ be a sequence (of complex numbers of modulus 1) whose closure is dense in the spectrum of U, in which each term occurs infinitely often. Apply Lemma 2 to write

$$S \sim D \oplus E \quad (\varepsilon/4), \tag{1}$$

where D is a diagonal operator with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$. (The unitary equivalence is, in fact, effected by the identity operator in this case, but nothing is lost by forgetting that.) In Lemma 2, to be sure, the shift was assumed to have multiplicity 1. The lemma is, however, applicable to shifts of all non-zero multiplicities; all that needs to be done is to break off a shift of multiplicity 1 as a direct summand, apply Lemma 2 as is, and then glue the fracture together again.

Let U^∞ be the direct sum of countably infinitely many copies of U; observe that U^∞ is unitary (and hence, in particular, normal) and that U^∞ is essential. Since D and $U^\infty \oplus D$ are essential unitary operators with the same spectrum, the remark following the proof of Lemma 4 shows that

$$D \sim U^\infty \oplus D \quad (\varepsilon/4). \tag{2}$$

Substitute (2) into (1) to get

$$S \sim U^\infty \oplus D \oplus E \quad (\varepsilon/2).$$

It follows that

$$U \oplus S \sim U \oplus U^\infty \oplus D \oplus E \quad (\varepsilon/2). \tag{3}$$

Since $U \oplus U^\infty$ is unitarily equivalent to U^∞, (3) implies

$$U \oplus S \sim U^\infty \oplus D \oplus E \quad (\varepsilon/2)$$

and hence, by (2)

$$U \oplus S \sim D \oplus E \quad (3\varepsilon/4). \tag{4}$$

Use (1) to replace the right side of (4) by S, and conclude that

$$U \oplus S \sim S \quad (\varepsilon).$$

The proof of Lemma 5 is complete.

Proof of the theorem. Suppose that V is an isometry of co-rank n ($1 \leqq n \leqq \infty$) on a separable Hilbert space, and suppose that $\varepsilon > 0$. Write V as $U \oplus S$, where U

is unitary and S is a unilateral shift (with, of course, multiplicity n) [6, Problem 118]. By Lemma 5

$$V \sim S \quad (\varepsilon).$$

Since an operator unitarily equivalent to a unilateral shift is a unilateral shift, this proves that in every ε neighborhood of V there is a unilateral shift (necessarily of the same co-rank as V), and the first half of the theorem follows.

The second half is proved similarly. Suppose that U is a unitary operator whose spectrum is the entire unit circle, and suppose that $\varepsilon > 0$. By Lemma 5

$$U \oplus S \sim S \quad (\varepsilon/2),$$

where S is a bilateral shift of multiplicity n. Since U and $U \oplus S$ are essential unitary operators with the same spectrum (here is where the hypothesis about the spectrum of U is used), it follows from the remark following the proof of Lemma 4 that

$$U \sim U \oplus S \quad (\varepsilon/2).$$

Consequence:

$$U \sim S \quad (\varepsilon),$$

and the proof is completed as in the unilateral case.

Scholium. *On a separable Hilbert space every isometry of non-zero co-rank is the sum of a pure isometry and an operator of arbitrarily small Hilbert—Schmidt norm.*

Except for the description of the size of the perturbation, this is the original version of the Douglas result mentioned after the statement of the theorem.

Experience shows that norm approximation theorems are likely to be difficult but worth the trouble; they give useful analytic insights into the behavior of operators. Strong and weak approximation theorems are usually easier to prove, but harder to find applications for. A comparison of the theorem proved above and the proposition below indicates that for approximation by shifts the customary situation prevails.

In what follows it is convenient to use the word "shift" ambiguously. A true statement and a valid proof result if it is interpreted consistently as either "unilateral shift" or "bilateral shift".

Proposition. *On a separable infinite-dimensional Hilbert space the strong closure of the set of shifts of multiplicity* 1 *is the set of all isometries; the weak closure of the set of shifts of multiplicity* 1 *is the set of all contractions.*

For the proof, consider first an arbitrary operator A on the given Hilbert space H, and a direct sum of the form $A \oplus B$ on $H \oplus H$. Assertion 1: if $f_1, \ldots, f_n$ are in H, then there exists an operator on H unitarily equivalent to $A \oplus B$ that agrees with

A on each f_j. To prove that, let V be an isometry from H onto $H \oplus H$ such that if f is in the (finite-dimensional) subspace spanned by $f_1, \ldots, f_n$, $Af_1, \ldots, Af_n$, then $Vf = [f, 0]$. (Here is where the infinite-dimensionality of H is used.) It follows that $V^*(A \oplus B)Vf_j = V^*(A \oplus B)[f_j, 0] = V^*[Af_j, 0] = Af_j$ for $j = 1, \ldots, n$.

Suppose now that U is an arbitrary unitary operator on H. Assertion 2: every strong neighborhood of U contains a shift of multiplicity 1. To prove this, consider a basic strong neighborhood of U, consisting of all operators T such that $\|Uf_j - Tf_j\| < \varepsilon$, $j = 1, \ldots, n$, where $f_1, \ldots, f_n$ are unit vectors in H and $\varepsilon > 0$. If S is a shift of multiplicity 1, then, by Assertion 1, there exists an operator unitarily equivalent to $U \oplus S$ that agrees with U on each f_j. Since, by Lemma 5, $U \oplus S \sim S$ (ε), it follows that some operator unitarily equivalent to S differs from U by less than ε on each f_j; this implies Assertion 2.

The preceding two paragraphs imply that the strong closure of the set of shifts of multiplicity 1 contains all unitary operators, and from this all else follows. Indeed, the *strong* closure of the set of unitary operators is known to be the set of all isometries [8, p. 892], and the *weak* closure of the set of unitary operators is known to be the set of all contractions [5, p. 128].

Problem. What are the answers to the corresponding questions for weighted shifts?

References

[1] S. K. Berberian, The Weyl spectrum of an operator, *Indiana U. Math. J.*, **20** (1970), 529—544.

[2] I. D. Berg, An extension of the Weyl—von Neumann theorem to normal operators, *Trans. Amer. Math. Soc.*, **160** (1971), 365—371.

[3] L. G. Brown, R. G. Douglas, and P. A. Fillmore. Unitary equivalence modulo compact operators and extensions of C^*-algebras. (Preprint, October 1972).

[4] N. Dunford and J. T. Schwartz, *Linear operators*. Part II, Interscience (New York, 1963).

[5] P. R. Halmos, Normal dilations and extensions of operators, *Summa Brasil. Math.*, **2** (1950), 125—134.

[6] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand (Princeton, 1967).

[7] P. R. Halmos, Permutations of sequences and the Schröder—Bernstein theorem, *Proc. Amer. Math. Soc.*, **19** (1968), 509—510.

[8] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887—933.

[9] P. R. Halmos, Continuous functions of Hermitian operators, *Proc. Amer. Math. Soc.*, **31** (1972), 130—132.

[10] W. Sikonia, The von Neumann converse of Weyl's theorem, *Indiana U. Math. J.*, **21** (1971), 121—123.

[11] J. von Neumann, *Charakterisierung des Spektrums eines Integraloperators*, Hermann (Paris, 1935).

INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47401

(Received May 27, 1972)