Operators similar to contractions

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In honor of Professor Béla Szőkefalvi-Nagy on his 60th birthday

1. Introduction. The purpose of this note is to establish the following characterization of those operators (continuous linear transformation) on a Hilbert space $\mathfrak{H}$ which are similar to contraction operators, and to point out some consequences.

Theorem. *An operator $T:\mathfrak{H}\to\mathfrak{H}$ is similar to a contraction if, and only if, there exist a Hilbert space $\mathfrak{K}$ and operators $A:\mathfrak{H}\to\mathfrak{K}$, $C:\mathfrak{K}\to\mathfrak{K}$, and $B:\mathfrak{K}\to\mathfrak{H}$ such that C is a contraction on $\mathfrak{K}$ (that is, $\|C\|\leqq 1$), and*

$$\sum_{n=0}^{\infty}\|BC^nA-T^n\|^2<\infty. \tag{1}$$

In one direction, this statement is trivial: if T is similar to a contraction $C:\mathfrak{H}\to\mathfrak{H}$, i.e. $T=SCS^{-1}$ for some operator $S:\mathfrak{H}\to\mathfrak{H}$ with operator inverse S^{-1}, then, taking $\mathfrak{K}=\mathfrak{H}$, $A=S^{-1}$, and $B=S$, we see that $BC^nA=T^n$ $(n=0, 1, 2, \ldots)$ so in this case (1) certainly holds. The reverse implication (proved in section 3) has some content, however, and provides a general principle from which we can immediately derive several of the known criteria for similarity to a contraction.

2. Applications. Perhaps the most direct application of this type is to the result of B. Sz.-Nagy and C. Foiaş that every operator of class C_ϱ is similar to a contraction; indeed, the theorem above, and the construction used in its proof, were inspired in part by a study of the Sz.-Nagy—Foiaş result. Recall that an operator $T:\mathfrak{H}\to\mathfrak{H}$ is said to be of class C_ϱ (where ϱ is some positive real number) when there exists a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ and a unitary operator $U:\mathfrak{K}\to\mathfrak{K}$ such that

$$T^n=\varrho P_{\mathfrak{H}}U^n|\mathfrak{H} \qquad (n=1,2,3,\ldots). \tag{2}$$

If we let $A:\mathfrak{H}\to\mathfrak{K}$ be the identity (inclusion) map, let C be the (contraction) U, and let $B=\varrho P_{\mathfrak{H}}$, then the Theorem certainly applies, since every term in (1) vanishes

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except the one for which $n=0$; it follows that such an operator T is similar to a contraction. The original proof of Sz.-Nagy and Foiaş may be found in [9] or in the book [10, pp. 92—95]; it depends upon clever use of the special properties of the ϱ-dilation U.

A generalization of the C_ϱ classes has been proposed by H. Langer (see [10, p. 55]): given a non-negative operator A on $\mathfrak{H}$, T is said to be of class C_A whenever there is a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ and a unitary operator U on $\mathfrak{K}$ such that

$$T^n = A^{1/2} P_{\mathfrak{H}} U^n A^{1/2} \qquad (n = 1, 2, \ldots). \tag{3}$$

Since V. Istrăţescu has shown (see [5]) that $C_A \in C_{\|A\|}$, it follows by the theorem of Sz.-Nagy and Foiaş that such an operator T is similar to a contraction. Using our theorem it is just as easy to prove this directly by observing that (1) holds when we let the operator A of the theorem be $A^{1/2}$ (forgive the notation), let B be $A^{1/2} P_{\mathfrak{H}}$, and let C be the contraction U on $\mathfrak{K}$.

Our theorem also yields immediately the result of G.-C. Rota (see [7, Theorem 2]) that every operator T with spectral radius $\nu(T)<1$ is similar to a contraction. To see this, choose $\mathfrak{K}$, A and B as you please, and let $C=0$; then (1) reduces to

$$\|BA-I\|^2 + \sum_{n=1}^{\infty} \|T^n\|^2 < \infty, \tag{4}$$

and this follows directly from the spectral radius formula: $\lim_{n\to\infty} \|T^n\|^{1/n} = \nu(T) (<1)$. Rota's result may also be derived from the theorem of Sz.-Nagy and Foiaş discussed above since, as we pointed out in [4; see Theorem 5. 1], the inequality $\nu(T)<1$ implies that $T \in C_\varrho$ for all large enough values of ϱ.

Our remarks in Section 4 point out a role that maybe be played by the Theorem in establishing yet another sufficient condition for similarly to a contraction: T is similar to a contraction providing its characteristic function $\Theta_T(\lambda)$ is uniformly bounded on the unit disc $\{\lambda : |\lambda| < 1\}$. This condition is due to C. Davis and C. Foiaş (see [1]).

Finally, the present theorem is an extension of the lemma which occurs in our paper [3], where further applications of these ideas may be found. It is possible, by making appropriate constructions, to prove our theorem by reducing it to the special case handled by the lemma of [3]. However, it is more efficient to proceed directly, and we give below a self-contained proof.

3. Proof of the theorem. It remains to show that, given A, B, and contraction $\mathscr{C}$ such that (1) holds, T is similar to a contraction. Equivalently, we must construct an equivalent inner product norm $|\cdot|$ on $\mathfrak{H}$ such that $|T| \leqq 1$, that is, such that T is a contraction with respect to $|\cdot|$.

To this end, define $|h|$ for each $h \in \mathfrak{H}$ by the relation

$$|h|^2 = \inf \{\| \sum_{n=0}^{\infty} C^n A h_n\|^2 + \sum_{n=0}^{\infty} \|h_n\|^2 : \sum_{n=0}^{\infty} T^n h_n = h\}; \tag{5}$$

here it is understood that only a finite number of the h_n are different from 0. It is a simple matter, using the triangle inequality in l^2, to show that $|\cdot|$ is a seminorm on $\mathfrak{H}$. Since $h = T^0 + T\cdot 0 + T^2 \cdot 0 + \cdots$,

$$|h| \leqq (\|Ah\|^2 + \|h\|^2)^{1/2} \leqq (\|A\|^2 + 1)^{1/2} \|h\|. \tag{6}$$

On the other hand, if $\sum_{n=0}^{\infty} T^n h_n = h$, then

$$\begin{aligned} \|h\| &= \|B (\sum_{n=0}^{\infty} C^n A h_n) + \sum_{n=0}^{\infty} (T^n - BC^n A) h_n\| \leqq \\ &\leqq \|B\| \| \sum_{n=0}^{\infty} C^n A h_n\| + \sum_{n=0}^{\infty} \|T^n - BC^n A\| \|h_n\| \leqq \\ &\leqq \left(\|B\|^2 + \sum_{n=0}^{\infty} \|T^n - BC^n A\|^2\right)^{1/2} \left(\| \sum_{n=0}^{\infty} C^n A h_n\|^2 + \sum_{n=0}^{\infty} \|h_n\|^2\right)^{1/2}, \end{aligned} \tag{7}$$

using the Schwarz inequality in l^2. It follows from (5) and (7) that

$$\|h\| \leqq \left(\|B\|^2 + \sum_{n=0}^{\infty} \|T^n - BC^n A\|^2\right)^{1/2} |h|. \tag{8}$$

and the constant in this inequality is finite by (1). We now know (via (6) and (8)) that $|\cdot|$ is a norm on $\mathfrak{H}$ equivalent to the given norm $\|\cdot\|$. Moreover, if $\sum_{n=0}^{\infty} T^n h_n = h$, then $\sum_{n=1}^{\infty} T^n h_{n-1} = Th$, and

$$\begin{aligned} \| \sum_{n=1}^{\infty} C^n A h_{n-1}\|^2 + \sum_{n=1}^{\infty} \|h_{n-1}\|^2 &= \|C (\sum_{n=0}^{\infty} C^n A h_n)\|^2 + \sum_{n=0}^{\infty} \|h_n\|^2 \leqq \\ &\leqq \| \sum_{n=0}^{\infty} C^n A h_n\|^2 + \sum_{n=0}^{\infty} \|h_n\|^2, \end{aligned} \tag{9}$$

since C is a contraction. From (9) it follows that $|Th| \leqq |h|$ $(h \in \mathfrak{H})$ so that $|T| \leqq 1$.

It remains to show that $|\cdot|$ is an inner product norm. Recall the characterization of inner product norms due to P. Jordan and J. von Neumann (see [6]): a norm $|\cdot|$ on $\mathfrak{H}$ is an inner product norm if (and only if) the "parallelogram law" holds, that is,

$$|h+g|^2 + |h-g|^2 = 2(|h|^2 + |g|^2) \qquad (h, g \in \mathfrak{H}). \tag{10}$$

Now if $\alpha > 2(|h|^2+|g|^2)$, then there exist h_n, g_n such that $\sum_{n=0}^{\infty} T^n h_n = h$, $\sum_{n=0}^{\infty} T^n g_n = g$, and

$$\alpha > 2\left(\|\sum_0^{\infty} C^n A h_n\|^2 + \|\sum_0^{\infty} C^n A g_n\|^2 + \sum_0^{\infty} (\|h_n\|^2 + \|g_n\|^2)\right). \tag{11}$$

By the parallelogram law for the given norms in $\mathfrak{K}$ and $\mathfrak{H}$, the right-hand side of (11) can be replaced by

$$\|\sum_0^{\infty} C^n A(h_n+g_n)\|^2 + \|\sum_0^{\infty} C^n A(h_n-g_n)\|^2 + \sum_0^{\infty} (\|h_n+g_n\|^2 + \|h_n-g_n\|^2). \tag{12}$$

Since $\sum_{n=0}^{\infty} T^n(h_n \pm g_n) = h \pm g$, it follows that $\alpha > |h+g|^2+|h-g|^2$, and hence

$$|h+g|^2+|h-g|^2 \leqq 2(|h|^2+|g|^2) \qquad (h, g \in \mathfrak{H}). \tag{13}$$

Finally, (13) is equivalent to (10), since the reverse inequality follows from (13) upon replacing h by $h+g$ and g by $h-g$.

4. Remarks. Condition (1) of our Theorem involves the operators BC^nA-T^n for *all* n. To demonstrate that some such condition is necessary, it may be well to point out that for an *arbitrary* operator T and any finite N we can satisfy the equalities $BC^nA=T^n$ ($n=0, 1, 2, \ldots, N$), with A, B, and C as in the Theorem. In fact, as L. J. Wallen and J. S. Johnson have observed (see P. R. Halmos [2, p. 910]), we may assume that $\mathfrak{H}$ is a subspace of $\mathfrak{K}$, that A is simply the identity on $\mathfrak{H}$, that C is an isometry, and that B is a skew projection of $\mathfrak{K}$ onto $\mathfrak{H}$. We then have, for $n=0, 1, 2, \ldots, N$,

$$BC^n|\mathfrak{H} = T^n. \tag{14}$$

If, however, (14) holds for *all* $n=0, 1, 2, \ldots$, then it follows as a corollary of our Theorem that T must be similar to a contraction. This corollary may be used as an alternative to some of the arguments of Davis and Foiaş in establishing their subtle result concerning operators with bounded characteristic function (see [1]). We shall indicate briefly how this may be done.

Given an operator T on Hilbert space $\mathfrak{H}$, let $Q_T=|I-T^*T|^{1/2}$, $J_T=\operatorname{sgn}(I-T^*T)$, and let $\mathfrak{D}_T$ denote the closure in $\mathfrak{H}$ of the subspace $Q_T\mathfrak{H}$. The characteristic function of T is the following operator-valued function of the complex variable λ:

$$\Theta_T(\lambda) = \left(-TJ_T + \lambda Q_{T^*}(I-\lambda T^*)^{-1} Q_T\right)|\,\mathfrak{D}_T. \tag{15}$$

The assumption of Davis and Foiaş in [1] is that $\Theta_T(\lambda)$ is defined throughout the open unit disc (that is, the spectral radius $\nu(T) \leqq 1$) and that

$$\sup_{|\lambda|<1} \|\Theta_T(\lambda)\| < \infty. \tag{16}$$

Davis and Foiaş show that the condition (16) implies that T is similar to a contraction, and in the course of their proof they conclude that the J-isometric dilation U of T is power-bounded. That is, they show that $\sup_{n\geq 0}\|U^n\|<\infty$, where U is the operator defined on the Hilbert space

$$K_+ \; H\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T\oplus\dots \tag{17}$$

by the relation

$$U(h_0\oplus h_1\oplus h_2\oplus\cdots) = Th_0\oplus Q_T h_0\oplus h_1\oplus h_2\oplus\cdots. \tag{18}$$

Note that $P_{\mathfrak{H}}U^n|\mathfrak{H}=T^n$ ($n=0, 1, 2, \ldots$), where $P_{\mathfrak{H}}$ denotes orthogonal projection of $\mathfrak{K}_+$ onto $\mathfrak{H}$ (imbedded in $\mathfrak{K}_+$ as in (17)). It is easy to see from (18) that U is expansive; that is, $\|Uk\|\geq\|k\|$ ($k\in\mathfrak{K}_+$). Once it is established, then, that U is power-bounded, the well-known technique of B. SZ.-NAGY (see [8]) allows us to define a new, equivalent, inner product on $\mathfrak{K}_+$, with respect to which U is isometric. In this new geometry on $\mathfrak{K}_+$, $P_{\mathfrak{H}}$ is generally a skew projection (no longer orthogonal), but, with $B=P_{\mathfrak{H}}$ and $C=U$, (14) holds for all n. By the corollary, then, T is similar to a contraction. Thus, with a further change in the geometry of $\mathfrak{H}$, T becomes a contraction.

The approach to the Davis—Foiaş theorem outlined above emphasizes the fact that assuming the J-isometric dilation U of T to be power-bounded is in itself sufficient to ensure that T is similar to a contraction. This assumption may be expressed directly in terms of T: it is easy to verify that $\sup_{n\geq 0}\|U^n\|<\infty$ if, and only if,

$$\sup_{\|h\|\leq 1}\sum_{n=0}^{\infty}\|Q_T T^n h\|^2<\infty. \tag{19}$$

The arguments of Davis and Foiaş show that (16)⇒(19), but it appears that (19) may be a weaker condition (which still allows the conclusion that T is similar to a contraction). In any case, (19) (and hence (16)) is not a *necessary* condition for similarity to a contraction; if, for example, T is the operator on $\mathbf{C}^2$ corresponding to the matrix $\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$, then $T^2=I$ and $Q_T\neq 0$, so that (19) is certainly violated, but it is well known that any finite-dimensional, power-bounded operator is similar to a contraction.

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