Operator inequalities and related dilations

By W. MLAK in Krakow (Poland)

Dedicated to Professor Béla Szőkefalvi-Nagy on his 60th birthday

We deal in the present paper with inequalities $T(a)^* Z T(a) \leqq Z$ where $T(\cdot)$ is a semi-group of operators in a sense to be precised below and Z is a fixed positive operator. We show that to such inequalities there correspond a uniquely determined positive definite function. Now the dilation theory enters which makes it possible to give a more or less precise intrinsic characterization of several properties of involved operators Z and $T(a)$. The inequalities in question have been studied by direct methods in [2] and [8] for $T(\cdot)$ being a semi-group of powers of a fixed operator.

In all what follows we consider the complex Hilbert spaces with usual notation for inner products and norms. If S is such a space then $L(S)$ stands for the algebra of all linear bounded operators in S and I_S denotes the identity operator in S. To begin with we formulate the following lemma:

Lemma. *Let H be a Hilbert space. Suppose we are given a set A totally ordered by the relation "$\leqq$". Let $Z \in L(H)$ be a positive operator. Assume that the function $T(\cdot, \cdot): A \times A \to L(H)$ satisfies the following conditions:*

(1) $T(a, a) = I_H$ *for* $a \in A$.

(2) $T(a, b) T(b, c) = T(a, c)$ *if* $c \leqq b \leqq a$.

(3) $T(a, b)$ *form a commutative family.*

Then, if

(4) $T(a, b)^* Z T(a, b) \leqq Z$ *for* $b \leqq a$

then the function

$$T(a, b) = \begin{cases} ZT(a,b) & \text{if} \quad b \leqq a, \\ T(b,a)^* Z & \text{if} \quad a \leqq b \end{cases}$$

is positive definite, i.e.,

$$\sum_{i,k} (T(a_i, a_k) f_i, f_k) \geqq 0$$

for every finite choice $a, \ldots, a_n \in A$, $f, \ldots, f_n \in H$.

The proof of the lemma may be performed exactly in the same way as that of Th. 2 of [5] by using Halperin's factoring method. It is also possible to apply directly Th. 2 [5] when using the semi-inner product $\langle f, g\rangle = (Zf, g)$ $(f, g \in H)$ (see comments after Theorem 2 below and [4]).

Suppose G is an additive subgroup of reals and let $G_+ = \{a \in G | a \geqq 0\}$. The semi-group $T(\cdot)$ on G_+ is a function $T(\cdot): G_+ \to L(H)$ such that $T(0) = I_H$ and $T(a+b) = T(a)T(b)$ for $a, b \in G_+$. Applying Lemma to the function $T(a, b) = T(a-b)$ $(a \geqq b \geqq 0,\ a, b \in G)$ we infer that if for $Z \in L(H)$, $Z \geqq O$

$$T(a)^* Z T(a) \leqq Z \quad \text{for} \quad a \in G_+ \tag{5}$$

then the function

$$T(a) = \begin{cases} ZT(a) & \text{if} \quad a \in G_+, \\ T(-a)^* Z & \text{if} \quad (-a) \in G_+ \end{cases}$$

is positive definite on G. By a suitable dilation theorem ([1], [7]) we get therefore a generalization of the celebrated theorem of Sz.-Nagy on unitary dilations of contractions:

Theorem 1. *Suppose the semi-group* $T(\cdot)$ *satisfies* (5). *Then there is a Hilbert space* K *and a unitary representation* $S(\cdot): G \to L(K)$ *and an operator* $R: H \to K$ *such that*

$$ZT(a) = R^* S(a) R \quad \textit{for} \quad a \in G_+. \tag{6}$$

The space K, *the operator* R *and the unitary group are determined uniquely up to equivalence by the minimality condition* $K = \bigvee_{a \in G} S(a)RH$.

If the minimality condition holds true then $S(\cdot)$ is called the minimal Z-dilation of $T(\cdot)$ and (6) the canonical representation for $T(\cdot)$.

Assume now that (5) holds true and let $S(\cdot)$ be the minimal Z-dilation of $T(\cdot)$. We define

$$M_- = \bigvee_{a \in G_+} S(-a) RH, \quad S_+(a) = S(-a) | M_- \quad (a \in G_+). \tag{7}$$

If $f, g \in H$ then for $a \in G_+$, $(-b) \in G_+$ we have

$$(RT(a)f, S(b)Rg) = (ZT(a-b)f, g) = (S_+(a)^* Rf, S(b)Rg).$$

Since the vectors $S(b)Rg$ $((-b) \in G_+, g \in H)$ span M_-, we conclude that the following theorem holds true:

Theorem 2. *Suppose that the semi-group $T(\cdot)$ satisfies* (5). *Let $S(\cdot)$ be the minimal Z-dilation of $T(\cdot)$ and let M_- and $S_+(\cdot)$ be defined by* (7). *Then $R_-: H\to M_-$ defined by $R_- f = Rf$ for $f\in H$ satisfies the following conditions:*

(8) $$R_- T(a) = S_+(a)^* R_- \quad \textit{for} \quad a\in G_+.$$

(9) $$Z = R_-^* R_-.$$

The above theorem includes as particular cases the Prop. 5. 1 of [8] p. 210 and Th. 5 of [2]. Notice that we do not require $T(\cdot)$ to be contractive.

The study of minimal Z-dilations may be reduced within certain limits to the study of ordinary dilations i.e. that ones for which $Z=I_H$. This is shown by arguments developed below, which, when suitably rearranged may stand for a direct proof of Theorem 1 without any appeal to Lemma. Suppose just that (5) holds true and let $S(\cdot)$ be the minimal Z-dilation of $T(\cdot)$.

Define $Q=\sqrt{Z}$, $H_1=\overline{R(Z)}=\overline{R(Q)}$. The relation $\tilde{T}(a)Qf=QT(a)f$ $(f\in H)$ determines a well defined semi-group $\tilde{T}(\cdot)$ of contractions in $L(H_1)$. It follows — see [4] — that $\tilde{T}(\cdot)$ has an ordinary minimal unitary dilation $U(a)$. Consequently $(U(a)Qf, Qg)=(\tilde{T}(a)Qf, Qg)=(ZT(a)f, g)=(S(a)Rf, Rg)$ for $a\in G_+$, $f, g\in H$; which implies that $U(\cdot)$ and $S(\cdot)$ are unitarily equivalent.

Suppose now that the operators $Z_1, Z_2\in L(H)$ are positive and

(10) $$T(a)^* Z_i T(a) \leqq Z_i \quad \text{for} \quad i=1,2, \quad a\in G_+$$

and the difference $\Delta Z = Z_2 - Z_1 \geqq O$ also satisfies the inequality

(11) $$T(a)^* \Delta Z T(a) \leqq \Delta Z \quad \text{for} \quad a\in G_+.$$

Let $Z_i T(a) = R_i^* S_i(a) R_i$ $(i=1, 2)$ be the canonical expression and K_i the minimal dilation space corresponding to Z_i. Following the arguments developed in [1], Lemma 4. 1 we conclude first from (11) that

$$\left\|\sum_{i|1}^{n} S_1(a_i) R_1 f_i\right\|^2 \leqq \left\|\sum_{i|1}^{n} S_2(a_i) R_2 f_i\right\|^2$$

for $a_i\in G$, $f_i\in H$. It follows that there is unique contraction $T: K_2\to K_1$ such that $TS_2(a)R_2 f = S_1(a)R_1 f$ for $a\in G$ and $f\in H$. Since the things are going about minimal dilations, the last equality yields that $TS_2(a)=S_1(a)T$ for all $a\in G$. We have just proved the following theorem:

Theorem 3. *Suppose that Z_1 and Z_2 satisfy* (10) *and* (11). *Then there exists a unique contraction $T: K_2\to K_1$ such that $TR_2=R_1$ and $TS_2(a)=S_1(a)T$ for all $a\in G$.*

Next we describe briefly some properties of polynomially bounded operators. We say that the operator $B \in L(H)$ is polynomially bounded if

$$\left\| \sum_{k|0}^{n} a_k B^k \right\| \leqq M \sup_{|z|=1} \left| \sum_{k|0}^{n} a_k z^k \right|$$

for every polynomial $\sum_{k|0}^{n} a_k z^k$ and with some finite M. If B is polynomially bounded then there are (so called elementary) measures $p(f, g)$ $(f, g \in H)$ on the unit circle C such that $\|p(f, g)\| \leqq M \|f\| \|g\|$ and

$$(12) \qquad (B^n f, g) = \int_C z^n dp(f, g) \qquad (n = 0, 1, 2 \dots)$$

for all $f, g \in H$. This is an easy consequence of results of [6] that then $H = H_a + H_s$, $B = B_a + B_s$ (both sums direct), $B_a \in L(H_a)$, $B_s \in L(H_s)$ and B_a, B_s are polynomially bounded and such that

$$(B_a^n f, g) = \int z^n dp^a(f, g) \qquad (f, g \in H_a;\ \ n = 0, 1, \dots),$$

$$(B_s^n f, g) = \int z^n dp^s(f, g) \qquad (f, g \in H_s;\ \ n = 0, 1, \dots),$$

where the elementary measures p^a and p^s satisfy the conditions:

$$(13) \qquad p^a(f, g) \ll m \quad \text{for} \quad f, g \in H_a,$$

$$(14) \qquad p^s(f, g) \perp m \quad \text{for} \quad f, g \in H_s.$$

$$(15) \qquad p(f, g) = p(f_a, g_a) + p(f_s, g_s),$$

m stands here for the normalized Lebesgue measure on C and f_a, g_a and f_s, g_s stand for projections of f, g on H_a and H_s respectively. One can show that B_s is similar to a unitary operator with singular spectrum. If B is a contraction then the above decompositions are orthogonal and B_s is unitary and singular. If $B = B_a$ $(B = B_s)$ then we say that B is m-continuous (m-singular respectively). The decomposition $B = B_a + B_s$ is called the Lebesgue decomposition of B.

Suppose that $Z \geqq O$ and $T \in L(H)$ satisfy the inequality

$$(16) \qquad T^* Z T \leqq Z.$$

Then for $H_1 = \overline{R(Z)}$, $Q = \sqrt{Z}$ the formula $\tilde{T}Qf = QTf$ $(f \in H)$ defines a contraction $\tilde{T} \in L(H_1)$. Let $H_1 = H_1^a \oplus H_1^s$, $\tilde{T} = \tilde{T}_a \oplus \tilde{T}_s$ be the corresponding Lebesgue decomposition of $\tilde{T}$. $\tilde{T}_s$ is unitary and singular. We now define Z_a, $Z_s \in L(H)$ by the formula

$$Z_a f = \hat{Q} P_a Q f, \quad Z_s f = \hat{Q} P_s Q f \qquad (f \in H),$$

where P_a and P_s are projections within H_1 on H_1^a and H_1^s respectively and $\tilde{Q}$ equals the restriction of Q to H_1. Since $P_a+P_s = I_{H_1}$ then

$$((Z_a+Z_s)f,f) = (Q(P_a+P_s)Qf,f) = (Zf,f)$$

for $f\in H$ i.e. $Z = Z_a+Z_s$. On the other hand $P_aQTf=\tilde{T}_aP_aQf$ for $f\in H$ and $\|\tilde{T}_a\|\leqq 1$ which implies that $\|P_aQTf\|^2\leqq\|P_aQf\|^2$ i.e. $T^*Z_aT\leqq Z_a$. By similar token, since T_s is unitary we get that $T^*Z_sT=Z_s$. Let $ZT^n=R^*V^nR$, $Z_aT^n=R_a^*V_{(a)}^nR_a$, $Z_sT^n= =R_s^*V_{(s)}^nR_s$ $(n\geqq 0)$ be the canonical expressions for positive definite functions related to Z, Z_a, Z_s according to Theorem 1, V, $V_{(a)}$ and $V_{(s)}$ being the corresponding unitary operators. Let F be the semi-spectral measure of $\tilde{T}$ and $F = F^a+F^s$ its Lebesgue decomposition relative to m. Then for $f\in H$, $n\geqq 0$,

$$(Z_aT^n,f,f)=(QP_aQT^nf,f)=(\tilde{T}_a^nQf,Qf)=\int z^n d(F^aQf,Qf)=\int z^n d(E_{(a)}R_af,R_af)$$

and

$$(Z_sT^nf,f)=\int z^n d(F^sQf,Qf)=\int z^n d(E_{(s)}R_sf,R_sf)$$

where $E_{(a)}$ and $E_{(s)}$ stand for spectral measure of $V_{(a)}$ and $V_{(s)}$ respectively. Since the disc algebra is a Dirichlet one on C we infer that

$$(F^aQf,Qf) = (E_{(a)}R_af,R_af)\ll m, \quad (F^sQf,Qf) = (E_{(s)}R_sf,R_sf)\perp m.$$

Consequently $V_{(a)}$ has a Lebesgue spectrum and $V_{(s)}$ is singular. On the other hand $V = V_a\oplus V_s$ (Lebesgue decomposition relative to m) and

$$\int z^n d(E_aRf,Rf)+\int z^n d(E_sRf,Rf) = (ZT^nf,f) = ((Z_a+Z_s)T^nf,f) =$$
$$= \int z^n d(E_{(a)}R_af,R_af)+\int z^n d(E_{(s)}R_sf,R_sf)$$

where E is the spectral measure of V, and $E = E_a\oplus E_s$ its Lebesgue decomposition. We conclude that for $f,g\in H$

$$(E_aRf,Rg) = (E_{(a)}R_af,R_af), \quad (E_sRf,Rg) = (E_{(s)}R_sf,R_sf)$$

which implies that for $n\geqq 0$

$$R_a^*V_{(a)}^nR_a = R^*V_a^nR = (P^aR)^*V_a^n(P^aR)$$
$$R_s^*V_{(s)}^nR_s = R^*V_s^nR = (P^sR)^*V_s^n(P^sR)$$

where $P^a=E_a(C)$, $P^s=E_s(C)$.

Summing up we get the following theorem:

Theorem 4. *Suppose T and Z satisfy* (16). *Then Z has a unique decomposition $Z = Z_a+Z_s$, $Z_a\geqq 0$, $Z_s\geqq 0$ where $T^*Z_aT\leqq Z_a$ $T^*Z_sT=Z_s$. The minimal Z_a (resp. Z_s) dilation of T is the m-continuous (resp. m-singular) part of the Z-dilation of T. Consequently, the minimal Z-dilation of T is an orthogonal sum of Z_a and Z_s dilations of T.*

Assume now that T which satisfies (16) is polynomially bounded. Let $p(f, g) = p^a(f, g)+p^s(f, g)$ be the Lebesgue decomposition (relative to m) of the elementary measure $p(f, g)$ of T. Using the previous notation for V, E_a, E_s we get for $n \geqq 0$, $f, g \in H$

$$(ZT^n f, g) = \int z^n d(E_a Rf, Rg) + \int z^n d(E_s Rf, Rg) = \\ = \int z^n dp^a(f, Zg) + \int z^n dp^s(f, Zg).$$

It follows now from the M. and F. Riesz theorem [3], Chapt. 4 that

$$\int z^n d(E_a Rf, Rg) = \int z^n dp^a(f, Zg), \quad (E_s Rf, Rg) \equiv p^s(f, Zg).$$

We conclude that for polynomially bounded T the following corollaries hold true:

Corollary 1. *If T is m-continuous then $Z=Z_a$ for every Z satisfying* (16), *i.e., every Z-dilation of T is m-continuous.*

Corollary 2. *If $Z=Z_a$ for T satisfying* (16) *then the range $R(Z)$ is included in the m-continuous part H_a of H of the Lebesgue decomposition related to T.*

Cor. 2 generalizes Cor. 5. 5 of [2]. Indeed, if $T^{*n}ZT^n \to O$ strongly then V is a bilateral shift with a complete wandering subspace equal to $\overline{(RT-VR)H}$. Consequently $V=V_a$. Notice that we infer Cor. 2 without using lifting of commutants.

References

[1] W. B. Arveson, Subalgebras of C*-algebras, *Acta Math.*, **123** (1969), 141—224.

[2] R. G. Douglas, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math.*, **30** (1969), 19—32.

[3] K. Hoffman, *Banach spaces of analytic functions* (Englewood Cliffs, N. Y., 1962).

[4] W. Mlak, Unitary dilations in case of ordered groups, *Ann. Pol. Math.*, **17** (1965), 321—328.

[5] _____ Positive definite contraction valued functions, *Bull. Acad. Polon. Sci. (sér. sci. math., astr. et phys.)*, **15** (1967), 509—512.

[6] _____ Decompositions and extensions of operator valued representations of function algebras, *Acta Sci. Math.*, **30** (1969), 181—193.

[7] B. Sz.-Nagy, Extensions of linear transformations in Hilbert spaces which extend beyond this space, Appendix to F. Riesz and B. Sz.-Nagy: *Functional analysis* (New York. 1960).

[8] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space* (Budapest—Amsterdam—London, 1970).

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
SECTION KRAKÓW

(Received March 10, 1972)