The order of magnitude of the Lebesgue functions and summability of function series

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Dedicated to Professor Béla Sz.-Nagy on his 60th birthday

1. Let X be a measurable space with a positive measure μ and let $F=\{f_k(x)\}$ $(k=0, 1, \ldots)$ be a sequence of L_μ-integrable functions on the set $E(\subset X)$ of positive measure. We shall consider the series

$$\sum_{k=0}^{\infty} c_k f_k(x) \tag{1}$$

with real coefficients satisfying

$$\sum_{k=0}^{\infty} c_k^2 < \infty \tag{2}$$

Let $T=(\alpha_{nk})$ $(n, k=0, 1, \ldots)$ be a doubly infinite matrix of real numbers determining a general summation process with the aid of the linear means

$$t_n(x) = \sum_{k=0}^{\infty} \alpha_{nk} c_k f_k(x).$$

We say that the series (1) is *T-summable* at the point $x(\in X)$ if the series defining $t_n(x)$ converges in the ordinary sense for all n (except perhaps finitely many of them) and the limit $\lim_{n\to\infty} t_n(x)$ exists at the point x in question.

Form the *Lebesgue functions* belonging to the sequence F of functions and to the summation process T as follows:

$$L_n(T, F; x) = \int_E |K_n(T, F; x, t)|\, d\mu(t),$$

where

$$K_n(T, F; x, t) = \sum_{k=0}^{\infty} \alpha_{nk} f_k(x) f_k(t).$$

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To avoid the unnecessary complications concerning the existence (in a certain sense) of $t_n(x)$ and $L_n(T, F; x)$, we shall consider the following two particular cases of summation processes T:

(i) If the functions $f_k(x)$ are assumed to be only L_μ-integrable on E, we shall confine ourselves to matrices T that have only finitely many nonzero elements in each row, i.e., which are such that $\alpha_{nk}=0$ for $k>k_n$ $(n=0, 1, \ldots)$.

(ii) If F is an orthonormal system defined on a set E of finite measure, then we shall only consider matrices T satisfying the condition

$$\sum_{k=0}^{\infty} \alpha_{nk}^2 < \infty \qquad (n = 0, 1, \ldots).$$

In this case, from (2) and this condition it immediately follows that $\sum_{k=0}^{\infty} \alpha_{nk}^2 c_k^2 < \infty$, and so we have by the Riesz—Fischer theorem that $t_n(x)$ is L_μ^2-integrable on E for every n. Furthermore, by virtue of

$$\sum_{k=0}^{\infty} \alpha_{nk}^2 \int_E f_k^2(x)\, d\mu(x) = \sum_{k=0}^{\infty} \alpha_{nk}^2 < \infty \qquad (n = 0, 1, \ldots),$$

and by B. Levi's theorem we can conclude, that $\sum_{k=0}^{\infty} \alpha_{nk}^2 f_k^2(x) < \infty$ for almost every x in E, and consequently $K_n(T, F; x, t)$ is L_μ^2-integrable on E as a function of t for almost every x in E and for every n. This implies, in particular, the existence of $L_n(T, F; x)$ for almost every x in E and for every n.

2. The order of magnitude of the Lebesgue functions may, in many cases, be a decisive factor in convergence problems.

In particular, taking

$$\alpha_{nk} = 1 - \frac{k}{n+1} \qquad (k = 0, 1, \ldots, n), \quad \alpha_{nk} = 0$$

$$(k = n+1, n+2, \ldots) \quad (n = 0, 1, \ldots),$$

we obtain the classical $(C, 1)$-summation process. Now we have

$$L_n((C, 1), F; x) = \int_E \left| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) f_k(x) f_k(t) \right| d\mu(t).$$

In this case G. ALEXITS and A. SHARMA [1] have proved the following theorems:

A. *Let F be a sequence of L_μ-integrable functions on a measurable set E of finite measure and let $\{\mu_n\}$ be a non-decreasing sequence of positive numbers. If $\sum c_k^2 < \infty$ and the condition $L_n((C, 1), F; x) = O(\mu_n)$ is uniformly satisfied on E, then the sums*

$$\sigma_n(x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) c_k f_k(x)$$

have the order of magnitude $O_x(\sqrt{\mu_n})$ on E almost everywhere.

B. *Let F be a sequence of L_μ-integrable functions on a measurable set E of finite measure satisfying the condition*

$$\int_E \left| \sum_{k=0}^{n} c_k d_k f_k(x) \right| d\mu(x) = O(1) \qquad (n = 0, 1, \ldots)$$

whenever $\sum c_k^2 d_k^2 < \infty$, and let $\{\mu_n\}$ be a non-decreasing sequence of positive numbers that is concave from below. Suppose that $L_n((C, 1), F; x) = O(\mu_n)$ for every $x \in E$ and

$$\sum_{k=0}^{\infty} c_k^2 \mu_k < \infty. \tag{3}$$

Then the series (1) *is* $(C, 1)$*-summable on E almost everywhere.*

They also remark that these results remain valid for any (C, α)-summation $(\alpha > 0)$ if we replace $L_n((C, 1), F; x)$ by the corresponding Lebesgue functions $L_n((C, \alpha), F; x)$.

We note that the above theorem for orthonormal systems is a well-known theorem of S. KACZMARZ [2]. G. SUNOUCHI [3] and L. LEINDLER [4] have extended Kaczmarz's theorem to the Riesz summation of orthogonal series. In this case

$$\alpha_{nk} = 1 - \frac{\lambda_k}{\lambda_{n+1}} \qquad (k = 0, 1, \ldots, n),$$

$$\alpha_{nk} = 0 \qquad (k = n+1, n+2, \ldots) \quad (n = 0, 1, \ldots),$$

and

$$L_n(R, F; x) = \int_E \left| \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right) f_k(x) f_k(t) \right| d\mu(t),$$

where $\{\lambda_n\}$ is a strictly increasing sequence of numbers with $\lambda_0 = 0$ and $\lambda_n \to \infty$ as $n \to \infty$.

To our knowledge, no analogous theorem for other summation processes has yet been proved. The following problem can be quite naturally raised: If for a summation process T and for an orthonormal system F defined on E the condition

$$L_n(T, F; x) = O(\mu_n)$$

is uniformly satisfied on E, is then the series (1), under condition (3), summable with respect to the concerning process almost everywhere in E?

A. V. EFIMOV [5] in the case of $\mu_n \to \infty$ $(n \to \infty)$, and K. TANDORI and the present author in a joint paper [6] in general, have essentially showed that the answer to this question is in the negative.

The aim of the present paper is to give a positive answer to the above question for a relatively large class of summation processes.

3. In the sequel we shall consider summation processes T with the following property: the estimate

$$\left| \sum_{k=0}^{\infty} \alpha_{nk} \alpha_{mk} f_k(x) f_k(y) \right| \leqq \sum_{i=0}^{\min(m,n)} \beta_i |K_i(T, F; x, y)| \tag{4}$$

holds for every m and n, where the positive numbers $\beta_i = \beta_i(T, F; \min(m, n))$ satisfy the inequalities

$$\sum_{i=0}^{\min(m,n)} \beta_i = O(1) \qquad (m, n = 0, 1, \ldots).$$

We note that if F is an orthonormal system defined on E, then the estimate (4) can be written in a more natural form as follows:

$$\left| \int_E K_n(T, F; x, t) K_m(T, F; y, t)\, d\mu(t) \right| = \left| \sum_{k=0}^{\infty} \alpha_{nk} \alpha_{mk} f_k(x) f_k(y) \right| \leqq$$

$$\leqq \sum_{i=0}^{\min(m,n)} \beta_i |K_i(T, F; x, y)|,$$

where the β_i's have the properties mentioned above.

We show that if T is the Riesz summation process defined by the sequence $\{\lambda_n\}$, then condition (4) is satisfied. (See also [4]). Supposing $n<m$, we obtain with the aid of the Abel transform that

$$\frac{1}{\lambda_{n+1}\lambda_{m+1}} \sum_{k=0}^{n} (\lambda_{n+1} - \lambda_k)(\lambda_{m+1} - \lambda_k) f_k(x) f_k(y) =$$

$$= \frac{\lambda_{m+1} - \lambda_{n+1}}{\lambda_{n+1}\lambda_{m+1}} \sum_{k=0}^{n} (\lambda_{n+1} - \lambda_k) f_k(x) f_k(y) + \frac{1}{\lambda_{n+1}\lambda_{m+1}} \sum_{k=0}^{n} (\lambda_{n+1} - \lambda_k)^2 f_k(x) f_k(y) =$$

$$= \frac{\lambda_{m+1} - \lambda_{n+1}}{\lambda_{m+1}} K_n(R, F; x, y) +$$

$$+ \frac{1}{\lambda_{n+1}\lambda_{m+1}} \sum_{k=0}^{n} (2\lambda_{n+1} - \lambda_k - \lambda_{k+1})(\lambda_{k+1} - \lambda_k) \sum_{l=0}^{k} f_l(x) f_l(y).$$

Substituting here $\lambda_{k+1} K_k(R, F; x, y) - \lambda_k K_{k-1}(R, F; x, y)$ for $(\lambda_{k+1} - \lambda_k) \sum_{l=0}^{k} f_l(x) f_l(y)$

($k=0, 1, \ldots, n$), a repeated Abel transform gives that the right-hand side can be written as

$$\frac{\lambda_{m+1}-\lambda_{n+1}}{\lambda_{m+1}} K_n(R,F;x,y)+\frac{1}{\lambda_{n+1}\lambda_{m+1}}\sum_{k=0}^{n-1}(\lambda_{k+2}-\lambda_k)\,\lambda_{k+1}K_k(R,F;x,y)+$$

$$+\frac{\lambda_{n+1}-\lambda_n}{\lambda_{m+1}} K_n(R,F;x,y)=\frac{\lambda_{m+1}-\lambda_n}{\lambda_{m+1}} K_n(R,F;x,y)+$$

$$+\frac{1}{\lambda_{n+1}\lambda_{m+1}}\sum_{k=0}^{n-1}(\lambda_{k+2}-\lambda_k)\,\lambda_{k+1}K_k(R,F;x,y).$$

Therefore, the estimate (4) holds for every n and m with $\beta_k=(\lambda_{k+2}-\lambda_k)/\lambda_{n+1}\geqq$ $\geqq(\lambda_{k+2}-\lambda_k)\lambda_{k+1}/\lambda_{n+1}\lambda_{m+1}$ ($k=0, 1, \ldots, n-1$) and $\beta_n=1\geqq(\lambda_{m+1}-\lambda_{n+1})/\lambda_{m+1}$, for which we have

$$\sum_{k=0}^{n}\beta_k=\frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}}+1\leqq 2 \qquad (n=0, 1, \ldots).$$

4. After these preliminaries our first result can be formulated as follows:

Theorem 1. *Let $F=\{f_k(x)\}$ be a sequence of L_μ-integrable functions on a measurable set E of finite measure, let c_k be a sequence of coefficients satisfying* (2), *and assume that the summation process T satisfies condition* (4). *If $\{\mu_n\}$ is a non-decreasing sequence of positive numbers for which the relation*

$$L_n(T,F;x)=O(\mu_n) \tag{5}$$

uniformly holds on E, then the estimate

$$t_n(x)=O_x(\sqrt{\mu_n})$$

holds almost everywhere in E.

The proof is a modification of the well-known proof of A. Kolmogoroff—G. Seliverstoff [7] and A. Plessner [8] for the trigonometric system, and of S. Kaczmarz [2] for arbitrary orthonormal systems.

We shall use an idea of C. J. Preston [9] which consists in a special representation of $t_n(x)$. Introduce an arbitrary orthonormal system $\{g_k(y)\}$ defined on a measure space Y with positive measure ν; then

$$t_n(x)=\int_Y \sum_{k=0}^{\infty}c_k g_k(t)\cdot\sum_{k=0}^{\infty}\alpha_{nk}f_k(x)g_k(t)\,d\nu(t).$$

Let $n(x)$ be the smallest index $\leqq n$ such that

$$\frac{t_{n(x)}(x)}{\sqrt{\mu_{n(x)}}}=\max_{0\leqq k\leqq n}\frac{t_k(x)}{\sqrt{\mu_k}}$$

holds. By Schwarz's inequality we have

$$I_n = \left| \int_E \frac{t_{n(x)}(x)}{\sqrt{\mu_{n(x)}}} d\mu(x) \right| \leqq \left\{ \int_Y \left[\sum_{k=0}^{\infty} c_k g_k(t) \right]^2 dv(t) \times \right.$$

$$\left. \times \int_Y \left[\int_E \frac{1}{\sqrt{\mu_{n(x)}}} \sum_{k=0}^{\infty} \alpha_{n(x),k} f_k(x) g_k(t) d\mu(x) \right]^2 dv(t) \right\}^{\frac{1}{2}} \leqq$$

$$\leqq \left\{ \sum_{k=0}^{\infty} c_k^2 \right\}^{\frac{1}{2}} \left\{ \int_E \int_E \int_Y \frac{1}{\sqrt{\mu_{n(x)}}} \sum_{k=0}^{\infty} \alpha_{n(x),k} f_k(x) g_k(t) \times \right.$$

$$\left. \times \frac{1}{\sqrt{\mu_{n(y)}}} \sum_{k=0}^{\infty} \alpha_{n(y),k} f_k(y) g_k(t) d\mu(x) d\mu(y) dv(t) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \int_E \frac{1}{\sqrt{\mu_{n(x)} \mu_{n(y)}}} \left| \sum_{k=0}^{\infty} \alpha_{n(x),k} \alpha_{n(y),k} f_k(x) f_k(y) \right| d\mu(x) d\mu(y) \right\}^{\frac{1}{2}}.$$

Using estimate (4) and the monotonity of $\{\mu_n\}$ we have

$$I_n = O(1) \left\{ \int_E \int_E \frac{1}{\sqrt{\mu_{n(x)} \mu_{n(y)}}} \sum_{i=0}^{\min(n(x),n(y))} \beta_i |K_i(T, F; x, y)| d\mu(x) d\mu(y) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \int_E \frac{1}{\mu_{n(x)}} \sum_{i=0}^{n(x)} \beta_i |K_i(T, F; x, y)| d\mu(x) d\mu(y) + \right.$$

$$\left. + \int_E \int_E \frac{1}{\mu_{n(y)}} \sum_{i=0}^{n(y)} \beta_i |K_i(T, F; x, y)| d\mu(x) d\mu(y) \right\}.$$

The validity of relation (5) on E implies the estimate

$$I_n = O(1) \left\{ \int_E \frac{1}{\mu_{n(x)}} \sum_{i=0}^{n(x)} \beta_i L_i(T, F; x) d\mu(x) + \int_E \frac{1}{\mu_{n(y)}} \sum_{i=0}^{n(y)} \beta_i L_i(T, F; y) d\mu(y) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \sum_{i=0}^{n(x)} \beta_i d\mu(x) + \int_E \sum_{i=0}^{n(y)} \beta_i d\mu(y) \right\}^{\frac{1}{2}} = O(1).$$

Since the sequence $\{t_{n(x)}(x)/\sqrt{\mu_{n(x)}}\}$ is increasing, it follows by B. Levi's theorem that

$$\frac{t_{n(x)}(x)}{\sqrt{\mu_{n(x)}}} < \infty$$

almost everywhere in E. The same is true for the sequence $\{-t_{n(x)}(x)/\sqrt{\mu_{n(x)}}\}$; hence

$$\frac{t_{n(x)}(x)}{\sqrt{\mu_{n(x)}}} = O_x(1)$$

almost everywhere, which implies our statement.

5. We need the following auxiliary result:

Lemma. *Let $\{\mu_n\}$ be a non-decreasing sequence of positive numbers. Let F be a sequence of L_μ-integrable functions on the set E and let $\{n_k\}$ be an increasing sequence of indices such that*

$$\int_E |s_n(x)|\, d\mu(x) = O(1) \tag{6}$$

and

$$s_{n_k}(x) = O_x(\sqrt{\mu_{n_k}}) \tag{7}$$

holds almost everywhere in E for every sequence of coefficients satisfying (2), *where $s_n(x)$ is the nth partial sum of the series* (1). *Then condition* (3) *implies the convergence of the partial sums $s_{n_k}(x)$ almost everywhere in E as $k \to \infty$.*

This lemma is contained in the cited paper of G. ALEXITS and A. SHARMA [1]. (See there Theorem 3.) We remark that (6) is trivially satisfied for orthonormal systems defined on a set E of finite measure.

In the sequel we suppose that the sequence F and the summation process T are such that there exists an increasing sequence n_k of indices for which the conditions

$$\text{(i)}\quad s_{n_k}(x) - t_{n_k}(x) = o_x(1) \quad \text{and} \quad \text{(ii)}\quad \max_{n_k < n \leqq n_{k+1}} |t_n(x) - t_{n_k}(x)| = o_x(1)$$

hold almost everywhere in E as $k \to \infty$ for every sequence of coefficients satisfying (2).

In particular, if F is an orthonormal system defined on a set E of finite measure and T is the Riesz summation process defined by $\{\lambda_n\}$, then the conditions (i) and (ii) are fulfilled by every sequence $\{n_k\}$ of indices for which

$$1 < q \leqq \frac{\lambda_{n_{k+1}}}{\lambda_{n_k}} \leqq r < \infty \quad (k = 1, 2, \ldots). \quad \text{(See A. ZYGMUND [10].)}$$

6. Now we are in a position to formulate our second result:

Theorem 2. *Suppose the sequence F of L_μ-integrable functions and the summation process T are such that there exists an increasing sequence $\{n_k\}$ of indices satisfying* (i) *and* (ii) *and such that condition* (6) *is also satisfied. If the inequality*

$$t_n(x) = O_x(\sqrt{\mu_n})$$

holds almost everywhere in E for every sequence of coefficients satisfying (2), *where* $\{\mu_n\}$ *is a non-decreasing sequence of positive numbers, then condition* (3) *implies the T-summability of the series* (1) *almost everywhere in E.*

In fact, by (i) we have that the inequality (7) holds almost everywhere in E for every sequence of coefficients satisfying (2). Applying our Lemma we get that condition (3) implies the convergence of the partial sums $\{s_{n_k}(x)\}$ almost everywhere in E. Using (i) and (ii) we obtain that, under (3), $\{t_n(x)\}$ converges almost everywhere in E, which entails our assertion.

7. Remarks. (i) It is clear that our Theorem 1 contains Theorem A as a special case. In particular, Theorem A remains valid for any Riesz summation process if we replace $L_n((C, 1), F; x)$ with the corresponding Lebesgue functions $L_n(R, F; x)$.

As for the orthonormal systems F, the results of G. SUNOUCHI [3] and L. LEINDLER [4] on the Riesz-summability of orthogonal series are also special cases of our Theorems 1 and 2.

(ii) We mention without proofs that Theorems 1 and 2 can be extended for other particular summation processes T such as, e.g., the de la Vallée Poussin summation, the Euler summation, etc.

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