On the consequences of permutation identities

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The aim of this note is to give a description of all permutation identities valid in a permutative semigroup [2]. YAMADA [4] was the first to consider permutation identities in semigroups. The best result in the field was attained by PERKINS [2] who proved that any commutative semigroup variety is finitely based. In the same work he gives an example showing that no similar proposition holds for varieties satisfying $xyzt = xzyt$. On the other hand, any permutative semigroup variety satisfying an identity of the form $x^{m+d} = x^m$ is finitely based. We give another class of (hereditary) finitely based varieties. As a matter of fact, this can be obtained from a result of PUTCHA and YAQUB [3] claiming that a semigroup in which a permutation identity of rather general type holds satisfies all permutation identities for products containing sufficiently many factors. From our results it would be easy to determine exactly the necessary number of factors, and to give a "standard" form of finite bases of identities (up to bases of permutation groups).

1. The consequence group

Following YAMADA [4], we call an identity of the form

$$(1) \qquad\qquad x_1 \dots x_n = x_{1\sigma} \dots x_{n\sigma}$$

a *permutation identity* if σ is a permutation of the set $\{1, \dots, n\}$. The number n will be called the *length* of identity (1).

Let $\mathfrak{S}$ be a semigroup variety. Denote the set of all permutation identities of length n which hold in $\mathfrak{S}$ by G_n and the set of the corresponding permutations by Γ_n. Obviously, Γ_n is a subgroup of the symmetric group Σ_n. The set of permutation identities of length $n+1$ which follow from G_n will be denoted by G'_n and the corresponding set of permutations by Γ'_n. Again, Γ'_n is a group called the *(first) consequence group of* Γ_n. The qth consequence group $\Gamma_n^{(q)}$ can be defined in a similar way

through the permutation identities of length $n+q$ which follow from G_n. We remark though trivial that $\Gamma_n^{(q_1+q_2)}=\Gamma_n^{(q_1)(q_2)}$.

Our main task consists in finding out how Γ_n' depends on Γ_n. For this purpose we shall first look for a comfortable system of generators of Γ_n'.

Suppose (1) holds in $\mathfrak{S}$. The subsequent $n+2$ identities follow immediately:

$$x_1 \ldots x_{n+1} = x_{1\sigma} \ldots x_{n\sigma} x_{n+1},$$

(2)
$$x_1 \ldots x_{n+1} = x_1 x_{1\sigma+1} \ldots x_{n\sigma+1},$$

$$x_1 \ldots x_{n+1} \equiv u_1^{(i)} \ldots u_n^{(i)} = u_{1\sigma}^{(i)} \ldots u_{n\sigma}^{(i)} \qquad (i=1, \ldots n)$$

where

$$u_j^{(i)} = \begin{cases} x_j & \text{if } j < i, \\ x_i x_{i+1} & \text{if } j=i, \\ x_{j+1} & \text{if } j > i. \end{cases}$$

The corresponding elements $\sigma', \sigma'', \lambda_1, \ldots, \lambda_n$ of the consequence group are given by the equations

(3$_1$)
$$j\sigma' = \begin{cases} j\sigma & \text{if } j \leq n, \\ n+1 & \text{if } j=n+1 \end{cases}$$

(3$_2$)
$$j\sigma'' = \begin{cases} 1 & \text{if } j=1, \\ (j-1)\sigma+1 & \text{if } 2 \leq j \leq n+1; \end{cases}$$

(3$_3$)
$$j\lambda_i = \begin{cases} j\sigma & \text{if } j \leq i\sigma^{-1}, \ j\sigma \leq i, \\ j\sigma+1 & \text{if } j < i\sigma^{-1}, \ j\sigma > i, \\ (j-1)\sigma & \text{if } j > i\sigma^{-1}, \ (j-1)\sigma < i, \\ (j-1)\sigma+1 & \text{if } j > i\sigma^{-1}, \ (j-1)\sigma \geq i \end{cases}$$

for $i=1, \ldots, n$.

Lemma 1. *The consequence group Γ_n' of Γ_n is generated by the elements* (3$_1$), (3$_2$), (3$_3$) *where σ ranges over Γ_n.*

It suffices to show that all identities in G_n' are consequences of the identities (2) where σ ranges over Γ_n. Now let

(4)
$$x_1 \ldots x_{n+1} = x_{1\tau} \ldots x_{(n+1)\tau}$$

be an identity in G_n', i.e. a consequence of G_n. This means that there exists a sequence of words $(x_1 \ldots x_{n+1} \equiv) a_0, a_1, \ldots, a_k (\equiv x_{1\tau} \ldots x_{(n+1)\tau})$ such that $a_r \equiv b_r u_1^{(r)} \ldots u_n^{(r)} c_r$, $a_{r+1} \equiv b_r u_{1\sigma(r)}^{(r)} \ldots u_{n\sigma(r)}^{(r)} c_r$ where b_r, c_r are arbitrary and $u_j^{(r)}$ nonempty words, $\sigma(r) \in \Gamma_n$. Denote the length of the word y by $l(y)$. Then $l(a_r)=l(a_{r+1})$ for all $r<k$ and thus, by induction, $l(a_r)=n+1$. On the other hand $l(a_r)=l(b_r)+\sum_{j=1}^{n} l(u_j^{(r)})+l(c_r)$, and, since $l(u_j^{(r)})>0$, there are only three possibilities: 1) $l(b_r)=0$, $l(c_r)=1$, $l(u_1^{(r)})= \cdots$

$\cdots = l(u_n^{(r)}) = 1$ and $a_r = a_{r+1}$ follows from an identity of type (2_1); 2) $l(b_r) = 1$, $l(c_r) = 0$,. $l(u_1^{(r)}) = \cdots = l(u_n^{(r)}) = 1$ and $a_r = a_{r+1}$ follows from an identity of type (2_2); 3) $l(b_r) = = l(c_r) = 0$, $l(u_i^{(r)}) = 2$ for exactly one i, $l(u_j^{(r)}) = 1$ for $j \neq i$ and $a_r = a_{r+1}$ follows from one of the identities (2_3), q.e.d.

The permutations (3) are not very easy to handle, therefore we shall use the system σ', $\lambda_i \lambda_{i+1}^{-1}$ $(1 \leq i \leq n-1)$, $\lambda_n \sigma'^{-1}$, $\sigma'' \lambda_1^{-1}$, equivalent to (3), instead. Introduce the notation

$$\gamma(i, j) = \begin{cases} (i\ i+1 \dots j) & \text{if } i \leq j, \\ (i\ i-1 \dots j) & \text{if } i > j. \end{cases}$$

Thus, $\gamma(j, i) = \gamma(i, j)^{-1}$. It is straightforward to check the formulae

(5)
$$\lambda_i = \gamma(n+1, i\sigma^{-1}) \sigma' \gamma(i, n+1) \quad \text{for } 1 \leq i \leq n,$$
$$\sigma'' = \gamma(n+1, 1) \sigma' \gamma(1, n+1).$$

Hence

(6) $\quad \lambda_i \lambda_{i+1}^{-1} = \gamma(n+1, i\sigma^{-1}) \sigma' \gamma(i, n+1) \gamma(n+1, i+1) \sigma'^{-1} \gamma((i+1)\sigma^{-1}, n+1) =$
$\qquad = \gamma(n+1, i\sigma^{-1}) \sigma' \cdot (i\ n+1) \cdot \sigma'^{-1} \gamma((i+1)\sigma^{-1}, n+1) =$
$\qquad = \gamma(n+1, i\sigma^{-1}) \cdot (i\sigma^{-1}\ n+1) \cdot \gamma((i+1)\sigma^{-1}, n+1) =$
$\qquad = \gamma(n+1, i\sigma^{-1}+1) \gamma((i+1)\sigma^{-1}, n+1) = \gamma((i+1)\sigma^{-1}, i\sigma^{-1}+1).$

for $1 \leq i \leq n-1$ and

(6') $\quad \lambda_n \sigma'^{-1} = \gamma(n+1, n\sigma^{-1}) \sigma' \gamma(n, n+1) \sigma'^{-1} = \gamma(n+1, n\sigma^{-1}) \cdot (n\sigma^{-1}\ n+1) =$
$\qquad\qquad = \gamma(n+1, n\sigma^{-1}+1),$

(6'') $\quad \sigma'' \lambda_1^{-1} = \gamma(n+1, 1) \sigma' \gamma(1, n+1) \gamma(n+1, 1) \sigma'^{-1} \gamma(1\sigma^{-1}, n+1) = \gamma(1\sigma^{-1}, 1).$

Remark that, by (3_1), (3_2) and (5),

$$((i+1)\sigma^{-1}) \sigma' \sigma''^{-1} = i\sigma^{-1}+1 \quad \text{for } 1 \leq i \leq n-1,$$
(7)
$$(n+1) \sigma' \sigma''^{-1} = n\sigma^{-1}+1,$$
$$(1\sigma^{-1}) \sigma' \sigma''^{-1} = 1,$$

and, since the symbols $n+1$, $1\sigma^{-1}$, $(i+1)\sigma^{-1}$ $(1 \leq i \leq n-1)$ are exactly the integers $1, \dots, n+1$ in a different order, we have obtained

Lemma 2. Γ_n' *is generated by the elements* σ', $\gamma(i, i\sigma' \sigma''^{-1})$ $(i = 1, \dots, n+1)$ *where* σ *ranges over* Γ_n.

The subgroup of Γ_n' generated by the cycles $\gamma(i, i\sigma' \sigma''^{-1})$ $(1 \leq i \leq n+1)$ will be denoted by Γ_n^*. As a generalization of Lemma 2, we have

Lemma 2'. *If*

(8)
$$\sigma' \sigma''^{-1} = v_1 \dots v_s$$

is the decomposition of $\sigma' \sigma''^{-1}$ into disjoint cycles for some $\sigma \in \Gamma_n$ and i, j occur in the same v_t, then $\gamma(i, j) \in \Gamma_n^*$.

Indeed, for some power of $\sigma' \sigma''^{-1}$ we have $i(\sigma' \sigma''^{-1})^c = j$. If $c = 1$ then $\gamma(i, j) \in \Gamma_n^*$ by its definition. Now let $c > 1$ and suppose the assertion holds for $c - 1$. Put $i(\sigma' \sigma''^{-1})^{c-1} = k$; then $\gamma(i, k) \in \Gamma_n^*$, $\gamma(k, j) = \gamma(k, k\sigma' \sigma''^{-1}) \in \Gamma_n^*$ and hence $\gamma(i, j) = \gamma(k, j) \cdot \gamma(i, k) \in \Gamma_n^*$.

2. Consequence groups of *f*-irreducible groups

The following subgroups of the symmetric group Σ_n will take important roles in what follows (A_n denotes, as usual, the alternating group):

$$\Sigma_{n,k} = \{\sigma | i\sigma = i \text{ for } i \geq k\}, \qquad \bar{\Sigma}_{n,k} = \{\sigma | i\sigma = i \text{ for } i \leq k\}, \qquad \Phi_k = \Sigma_{n,k} \otimes \bar{\Sigma}_{n,k},$$

$$\Sigma_n^{(e)} = \{\sigma | i\sigma = i \text{ for odd } i\}, \qquad \Sigma_n^{(o)} = \{\sigma | i\sigma = i \text{ for even } i\},$$

$$\Sigma_n^{(p)} = \Sigma_n^{(e)} \otimes \Sigma_n^{(o)} = \{\sigma | i\sigma \equiv i \,(\text{mod } 2)\}, \qquad A_n^{(p)} = \Sigma_n^{(p)} \cap A_n.$$

Observe that $\sigma \in \Phi_k$ iff the images $i\sigma$ of elements $i \leq k$ precede those of elements $i \geq k$ (in particular, $k\sigma = k$). Remark also $\Sigma_{n,n+1} = \bar{\Sigma}_{n,0} = \Sigma_n$.

The role of $A_n^{(p)}$ is clear from

Lemma 3. $\Gamma_n' \subseteq A_{n+1}$ *iff* $\Gamma_n \subseteq A_n^{(p)}$.

Proof. If $\Gamma_n \subseteq A_n^{(p)}$ then σ', σ'' and $\sigma' \sigma''^{-1}$ are contained in $A_{n+1}^{(p)}$ for all $\sigma \in \Gamma_n$. Thus, $i \equiv i\sigma' \sigma''^{-1}$ (mod 2) for every i and $\gamma(i, i\sigma' \sigma''^{-1}) \in A_{n+1}$.

Conversely, suppose $\Gamma_n \nsubseteq A_n^{(p)}$ and let $\sigma \in \Gamma_n \setminus A_n^{(p)}$. If $\sigma \notin A_n$ then $\sigma' \notin A_{n+1}$. If $\sigma \notin \Sigma_n^{(p)}$ suppose i is the least natural number such that

(9)
$$i\sigma^{-1} \not\equiv i \quad (\text{mod } 2).$$

Then

(10)
$$\gamma(i\sigma^{-1}, (i\sigma^{-1})\sigma' \sigma''^{-1}) \notin A_{n+1}.$$

Indeed, for $i = 1$ we have $(1\sigma^{-1})\sigma' \sigma''^{-1} = 1$ and (10) follows from (9). If $i > 1$ then $(i\sigma^{-1})\sigma' \sigma''^{-1} = (i-1)\sigma^{-1} + 1 \equiv i - 1 + 1 = i$ (mod 2) and therefore $i\sigma^{-1} \not\equiv (i\sigma^{-1})\sigma' \sigma''^{-1}$ (mod 2) which proves (10).

The permutation group Γ_n will be called *fixelement-reducible* or *f-reducible* if $\Gamma_n \subseteq \Phi_k$ for some $k \leq n$, and *fixelement-irreducible (f-irreducible)* in the opposite case. Now we want to investigate the case where Γ_n is f-irreducible.

Lemma 4. *If Γ_n is f-irreducible then for every k $(1 \leq k \leq n)$ there exist symbols i, j such that $i \leq k < j$, $\gamma(i, j) \in \Gamma_n^*$.*

Proof. Since Γ_n is f-irreducible there exists $\sigma \in \Gamma_n \setminus \Phi_k$. If $k = 1$ this means $1\sigma \neq 1$, so that $1\sigma^{-1} \neq 1$ and, by virtue of (7_3) and Lemma 2, we have $\gamma(1, 1\sigma^{-1})$ $(= \gamma(1\sigma^{-1}, 1)^{-1}) \in \Gamma_n^*$. Now put $k \geq 2$. Then there exist elements i, l with $1 \leq i \leq k \leq$ $\leq l \leq n$, $i\sigma > l\sigma$ (and therefore $i \neq 1\sigma^{-1}$). It is easy to see that one can even suppose $l\sigma = i\sigma - 1$. Now by (7) $i\sigma' \sigma''^{-1} = (i\sigma - 1)\sigma^{-1} + 1 = l + 1$, so that $\gamma(i, l+1) \in \Gamma_n^*$, and the lemma is proved.

Corollary 2. *If Γ_n is f-irreducible then Γ_n^* is transitive.*

Indeed, $k\gamma(i, j) = k + 1$; thus, every symbol $(< n+1)$ can be carried over to every greater symbol and, taking into account the inverses of the γ's, it can be carried over to every element.

This corollary is majorized by the following lemma. The proof of the lemma, however, relies upon the corollary itself.

Lemma 5. *If Γ_n is f-irreducible then Γ_n^* is doubly transitive.*

Proof. Since Γ_n^* is already known to be transitive, we have to prove only that for every k $(1 < k < n+1)$ there exists a permutation $\varrho_k \in \Gamma_n^*$ such that

(11) $$k\varrho_k = k+1, \quad 1\varrho_k = 1.$$

Recall that $1\sigma^{-1} \neq 1, n+1$ for some $\sigma \in \Gamma_n$. If $\gamma(1, n+1) \in \Gamma_n^*$ then it has a power such that the permutation $\varrho_k = \gamma(n+1, 1)^{r_k} \cdot \gamma(1, 1\sigma^{-1})\gamma(1, n+1)^{r_k}$ satisfies (11) (for this, choose $k - 1\sigma^{-1} < r_k \leq \min(k-1, n+1-1\sigma^{-1})$). For the rest of the proof suppose $\gamma(1, n+1) \notin \Gamma_n^*$. By Lemma 4, there exist symbols l, m such that $l \leq k < m$, $\gamma(l, m) \in \Gamma_n^*$. If $1 < l$ put $\varrho_k = \gamma(l, m)$. If $l = 1$ then, by assumption, $m < n+1$. Thus, there exist i, j with $i \leq m < j$, $\gamma(i, j) \in \Gamma_n^*$. Put $\tau = \gamma(j, i)\gamma(1, m)\gamma(i, j)$ and

$$\varrho_k = \begin{cases} \tau \, (= \gamma(2, m+1)) & \text{if } i = 1, \\ \gamma(i, j) & \text{if } 1 < i \leq k, \\ \gamma(1, m)\tau^{-1} \, (= (k\,k+1)(m\,m+1)) & \text{if } i = k+1, \\ \tau^{-1}\gamma(1, m)\tau \, (= (i\,i+1)\gamma(2, m+1)(i\,i+1)) & \text{else.} \end{cases}$$

This proves our lemma.

Corollary 3. *If Γ_n is f-irreducible then Γ_n^* is primitive.*

Now we formulate the basic

Theorem 1. *If Γ_n is f-irreducible then its consequence group Γ_n' is 1) the subgroup Δ' of Σ_6 generated by $\gamma(1, 4)$ and $\gamma(3, 6)$ if $n = 5$, $\Gamma_5 = \Delta = \{(1), (14)(25)\}$; 2) A_{n+1} if $\Gamma_n \subseteq A_n^{(p)}$; 3) Σ_{n+1} else.*

Remark. $\varDelta'$ is a group isomorphic to Σ_5. It can be obtained from the subgroup $\Sigma_{6,6}$ of Σ_6 (having 6 for invariant symbol) by an outer automorphism of the latter one.]

Proof. The fact that the consequence group of $\varDelta$ is $\varDelta'$ can be checked by a straightforward calculation. Remark only that for $\sigma=(14)(25)$ we have $\sigma'\sigma''^{-1}=$ $=(14)(36)$ so that $\varGamma_n^*=\varDelta'$ and $\sigma'=\gamma(6,3)\gamma(1,4)^2\gamma(3,6)\in\varGamma_n^*$.

In virtue of Lemma 3, all we need to prove is that $\varDelta$ is the only f-irreducible group the consequence group of which does not contain the alternating group. In doing this we shall rely upon the following facts (see e.g. [1]):

I. If a subgroup $\varPi_{m-q}$ of Σ_m has q invariant symbols and is transitive and primitive on the rest then any primitive subgroup of Σ_m which contains $\varPi_{m-q}$ is $(q+1)$-fold transitive ([1], Theorem 5. 6. 2).

II. For $m>12$, $t>3\sqrt{m}-2$, the only t-fold transitive subgroups of Σ_m are Σ_m and A_m ([1], p. 68.).

III. If $m=kp+r$ where p is prime, $p>k$, $r>k$, $r>2$ then the only $(r+1)$-fold transitive subgroups of Σ_m are Σ_m and A_m ([1], Theorem 5. 7. 2).

Suppose $\varGamma_n$ is f-irreducible. If $\varGamma_n^*$ contains a transposition we have obviously $\varGamma_n^*=\Sigma_{n+1}$ (because of double transitivity). If $\varGamma_n^*$ contains an element of the form $\gamma(k-2,k)$ then $A_{n+1}\subseteq\varGamma_n^*$. Indeed, for $n=2$ the assertion is obvious. Let $n>2$. It suffices to show that if $k<n+1$ then $\gamma(k-1,k+1)\in\varGamma_n^*$ and if $k-2>1$ then $\gamma(k-3,k-1)\in\varGamma_n^*$ since these imply $\gamma(q-2,q)\in\varGamma_n^*$ for all $3\leqq q\leqq n+1$ and these cycles generate A_{n+1}. Let us prove the first part; the other one can be treated analogously. By Lemma 4, there exists a cycle $\gamma(i,j)\in\varGamma_n^*$ such that $i\leqq k\leqq j$. If $i\leqq k-2$ then $\gamma(j,i)\gamma(k-2,k)\gamma(i,j)=\gamma(k-1,k+1)\in\varGamma_n^*$. If $i=k-1$ or $i=k$ then $\gamma(j,i)\gamma(k-2,k)\gamma(i,j)=(k-2\,k\,k+1)$ or $(k-2\,k-1\,k+1)$, respectively. However, since $\gamma(k,k-2)\cdot(k-2\,k\,k+1)\cdot\gamma(k-2,k)=(k+1\,k-1\,k-2)$, in both cases we have $(k-1\,k\,k+1)=(k+1\,k-1\,k-2)(k-2\,k\,k+1)\in\varGamma_n^*$.

Now consider two different cycles $\gamma(i_1,j_1)$, $\gamma(i_2,j_2)\in\varGamma_n^*$ (e.g. $\gamma(1,1\sigma^{-1})$ for $1\sigma\neq1$ and $\gamma(n\tau^{-1}+1,n+1)$ for $n\tau\neq n$). We may suppose $i_1<j_1$, $i_1\leqq i_2<j_2$. If, moreover, $j_2<j_1$ then $\gamma(i_2-1,j_2-1)=\gamma(i_1,j_1)\gamma(i_2,j_2)\gamma(j_1,i_1)\in\varGamma_n^*$. If $j_2=j_1$ then $\gamma(i_2+1,j_2+1)=\gamma(j_1,i_1)\gamma(i_2,j_2)\gamma(i_1,j_1)\in\varGamma_n^*$. In both cases we have two cycles of the form $\gamma(k,l)$ and $\gamma(k-1,l-1)$. However then $\gamma(l,k)\gamma(l-1,k-1)\gamma(k,l)^2=$ $=\gamma(k-1,k+1)\in\varGamma_n^*$ and hence $\varGamma_n^*\supseteq A_{n+1}$.

The case $i_1=i_2$, $j_1<j_2$ is symmetrical to the second subcase of the above one.

If $j_1=i_2$ then $\gamma(i_2,j_2)\gamma(i_1,j_1)=\gamma(i_1,j_2)$ and the pair $\gamma(i_1,j_2)$, $\gamma(i_2,j_2)$ gives the first case again.

If $j_1<i_2$ and $n\leqq6$ then at least one of both cycles is of length $\leqq3$ and the former argument yields $\varGamma_n^*\supseteq A_{n+1}$ once more. Let $n>6$ (i.e. $n+1\geqq8$). One of both cycles is of length $\leqq\dfrac{n+1}{2}$; denote this one by $\gamma(i,j)$. If $j<n+1$ take $\gamma(k,l)\in\varGamma_n^*$ with

$k \leq j < l$ (if $j = n+1$ then $i > 1$ and we ought to demand $l < i-1 \leq k$). The permutations $\gamma(i,j)$, $\gamma(l,k)\gamma(i,j)\gamma(k,l)$ generate a subgroup Π_{j-i+2} of Γ_n^* having at least $\frac{n-1}{2}$ invariant symbols and being primitive on the rest. Thus, by I, Γ_n^* is $\frac{n+1}{2}$-fold transitive. If $n \geq 27$ it holds $\frac{n+1}{2} > 3\sqrt{n+1} - 2$ and $\Gamma_n^* \supseteq A_{n+1}$ follows from II.

For $7 \leq n \leq 26$ III allows the following maximal multiplicities of transitivity:

$r = 3$ for $n+1 = 8, 9, 10, 13, 14, 16, 17, 20, 22, 25, 26;$
$r = 4$ for $n+1 = 11, 15, 18, 19, 21, 23, 27;$
$r = 5$ for $n+1 = 12, 24.$

In all cases $r < \frac{n+1}{2}$.

Thus, the only remaining possibility is $i_1 < i_2 < j_1 < j_2$ for every pair of cycles. Suppose there are three different cycles $\gamma(i_1, j_1)$, $\gamma(i_2, j_2)$, $\gamma(i_3, j_3) \in \Gamma_n^*$, $i_1 < i_2 < i_3 < < i_1 < j_2 < j_3$. Then

$$\varrho_1 = \gamma(i_1, j_1)\gamma(j_3, i_3)\gamma(j_1, i_1)\gamma(i_3, j_3) = (i_3 - 1\, i_3)(j_1 j_1 + 1) \in \Gamma_n^*,$$

$$\varrho_2 = \gamma(i_2, j_2)\gamma(j_3, i_3)\gamma(j_2, i_2)\gamma(i_3, j_3) = (i_3 - 1\, i_3)(j_2 j_2 + 1) \in \Gamma_n^*,$$

$$\varrho_1 \varrho_2 = (j_1 j_1 + 1)(j_2 j_2 + 1) \in \Gamma_n^*,$$

$$\gamma(i_1, j_1)\varrho_2\varrho_1\gamma(j_1, i_1)\varrho_1\varrho_2 = \gamma(j_1 - 1, j_1 + 1) \in \Gamma_n^*.$$

Thus, we may suppose Γ_n^* contains only two cycles: $\gamma_1 = \gamma(i_1, j_1)$ and $\gamma_2 = \gamma(i_2, j_2)$. However, this implies $\sigma'\sigma''^{-1} = (i_1 j_1)(i_2 j_2)$ or (1) for every $\sigma \in \Gamma_n$ (recall $\sigma'\sigma''^{-1} \in A_{n+1}$). But $\sigma'\sigma''^{-1}$ determines σ uniquely; indeed, σ' does and the conditions $\sigma'^{-1}\gamma(n+1, 1)\sigma' = \sigma'\sigma''^{-1}\gamma(n+1, 1)$, $(n+1)\sigma' = n+1$ determine σ'. Hence Γ_n is a two-element group and $\sigma(\neq 1)$ is of order 2. As Γ_n is f-irreducible, $1\sigma \neq 1$, $n\sigma \neq n$ and, consequently, $i_1 = 1$, $j_2 = n+1$, $j_1 = 1\sigma^{-1} = 1\sigma$, $i_2 = n\sigma + 1$ and $\sigma'\sigma''^{-1} = = (1\, j_1)...(i_2 - 1\, n)(2\, j_1 + 1)...(i_2\, n+1) = (1\, j_1)(i_2\, n+1)$. Hence one finds by a routine induction $\sigma = (1\, j_1)(2\, j_1 + 1)...(n - j_1 + 1\, n)$, $i_2 = n - j_1 + 2$. If $j_1 = n$ then $i_2 = 2$ and $\gamma_2^{-1}\gamma_1^{-1}\gamma_2^2 = (1\, 2\, 3)$. Now let $j_1 < n$, $i_2 > 2$. Form the following elements:

$$\alpha = \gamma_2^{-1}\gamma_1\gamma_2 = (1 ... i_2 - 1\, i_2 + 1 ... j_1 + 1),$$

$$\beta = \gamma_2^{-1}\gamma_1\gamma_2^2 = (1 ... i_2 - 1\, i_2 + 2 ... j_1 + 2),$$

$$\delta = \alpha\beta^{-1} = (i_2 - 1\, i_2 + 1)(j_1 + 1\, j_1 + 2),$$

$$\varepsilon = \delta^{-1}\gamma_1\delta\gamma_1^{-1} = (i_2 - 2\, i_2)(i_2 - 1\, i_2 + 1),$$

and, if $i_2 > 3$,

$$\zeta = \gamma_1\varepsilon\gamma_1^{-1} = (i_2 - 3\, i_2 - 1)(i_2 - 2\, i_2).$$

Then

$$\varepsilon\zeta = (i_2 - 3 \ i_2 - 1 \ i_2 + 1),$$

$$\gamma_2 \gamma_1 \gamma_2 \gamma_{\bar 1}^{\,-2} \gamma_2 \varepsilon\zeta \gamma_{\bar 2}^{\,-1} \gamma_1^2 \gamma_{\bar 2}^{\,-1} \gamma_{\bar 1}^{\,-1} \gamma_{\bar 2}^{\,-1} = (i_2 - 2 \ i_2 - 1 \ i_2) \quad \text{if} \quad i_2 = j_1 - 1,$$

$$\gamma_2 \gamma_{\bar 1}^{\,-2} \gamma_2 \varepsilon\zeta \gamma_{\bar 2}^{\,-2} \gamma_1^2 \gamma_{\bar 2}^{\,-1} = (i_2 - 1 \ i_2 \ i_2 + 1) \quad \text{else.}$$

On the other hand, if $i_2 = 3$, $j_1 > 4$ put $\eta = \delta^{-1} \gamma_{\bar 1}^{\,-1} \delta \gamma_1 = (24)(35)$. Then

$$\eta\varepsilon = (1 \ 3 \ 5),$$

$$\gamma_2 \gamma_{\bar 1}^{\,-1} \gamma_2 \gamma_{\bar 1}^{\,-2} \gamma_2 \gamma_1 \eta\varepsilon \gamma_{\bar 1}^{\,-1} \gamma_{\bar 2}^{\,-1} \gamma_1^2 \gamma_{\bar 2}^{\,-1} \gamma_1 \gamma_{\bar 2}^{\,-1} = (2 \ 3 \ 4).$$

Thus, we have reduced the problem to the case $i_2 = 3$, $j_1 = 4$ corresponding to $\Gamma_n = \Delta$. The theorem is proved.

From the proof it turns out that $\Gamma_n^* \supseteq A_{n+1}$ if $\Gamma_n \neq \Delta$ and $\Gamma_n^* = \Delta'$ if $\Gamma_n = \Delta$. Combining this with Lemma 3 and the plus information on Γ_n^* its proof comprises, we have

Colollary 4. *If Γ_n is f-irreducible then*

$$\Gamma_n^* = \begin{cases} \Delta' & \text{if} \quad \Gamma_n = \Delta, \\ A_{n+1} & \text{if} \quad \Gamma_n \subsetneqq \Sigma_n^{(p)}, \\ \Sigma_{n+1} & \text{else.} \end{cases}$$

Thus, $\Gamma_n^* = \Gamma_n'$ or else $\Gamma_n \subseteq \Sigma_n^{(p)}$, $\Gamma_n \nsubseteq A_n$.

3. The general case

The case of f-reducible groups can be reduced now to that of f-irreducible ones. Let $\Gamma_n = \bigcap\limits_{t=1}^{s} \Phi_{k_t}$ $(1 \leq k_1 < \cdots < k_s \leq n)$ and $\Gamma_n \nsubseteq \Phi_k$ for any other k. Put, furthermore, $k_0 = 0$, $k_{s+1} = n+1$, and denote $\Sigma_{n,k_t} \cap \bar\Sigma_{n,k_{t+1}}$ by P_t, $k_{t+1} - k_t - 1$ by j_t $(t = 0, \ldots, s)$. Then $\Gamma_n \subseteq \prod\limits_{t=0}^{s} P_t$ and it is easy to see that the mapping $\varphi_t : P_t \to \Sigma_{j_t}$ defined by $j(\varrho\varphi_t) = (j + k_t)\varrho - k_t$ is an isomorphism. Denote by Λ_t the projection of Γ_n in P_t. Then Γ_n is a subdirect product of $\Lambda_0, \ldots, \Lambda_s$. Obviously, $\Lambda_t \varphi_t$ is f-irreducible.

Every $\sigma \in \prod\limits_{t=0}^{s} P_t$ has a unique factorization

(12) $\sigma = \sigma_0 \ldots \sigma_s$ $(\sigma_t \in P_t)$;

in particular, $\sigma_t \in \Lambda_t$ if $\sigma \in \Gamma_n$. Analogously, put $P_t' = \Sigma_{n+1,k_t} \cap \bar\Sigma_{n+1,k_{t+1}+1} \subseteq \Sigma_{n+1}$; then every element of $\prod\limits_{t=0}^{s} P_t'$ has a unique factorization

(12′) $\tau = \tau_0 \ldots \tau_s$ $(\tau_t \in P_t')$.

Define $\varphi_t' : P_t' \to \Sigma_{j_t+1}$ by the same rule as we did φ_t (only for $i \leq j_t+1$ instead of $i \leq j_t$).

Those subdirect factors Λ_t contained in $\Sigma_n^{(p)}$ but not in $A_n^{(p)}$ behave in a manner slightly different from the rest. Introduce therefore the notation

$$T = \{t \,|\, 0 \leq t \leq s \wedge \Lambda_t \subseteq \Sigma_n^{(p)} \wedge \Lambda_t \nsubseteq A_n^{(p)}\}$$

and the projection $\mu : \Gamma_n \to \prod_{t \in T} \Lambda_t$ which maps σ onto $\prod_{t \in T} \sigma_t$.

The consequence group of an f-reducible group is now fully described by

Theorem 2. *Let* $\Gamma_n \subseteq \bigcap_{t=1}^{s} \Phi_{k_t}$ $(1 \leq k_1 < \cdots < k_s \leq n)$ *and* $\Gamma_n \nsubseteq \Phi_k$ *for any other* k. *Then*

(13)
$$\Gamma_n' = (\Gamma_n \mu)' \times \prod_{t \notin T} \Lambda_t'.$$

Furthermore, $\Lambda_t' = (\Lambda_t \varphi_t)' \varphi_t'^{-1}$ *for* $1 \leq t \leq s$ *and*

(14)
$$(\Gamma_n \mu)' = \left\{ \prod_{t \in T} \tau_t \,|\, \tau_t \in P_t' \wedge (\exists \sigma \in \Gamma_n) \, \forall_t (\tau_t \in A_{n+1} \Leftrightarrow \sigma_t \in A_n) \right\}.$$

Proof. Since $i\sigma = i\sigma_t$ for $k_t \leq i \leq \max(k_{t+1}, n)$, it follows $i\sigma' = i\sigma_t'$, $i\sigma'' = (i-1)\sigma + 1 = (i-1)\sigma_t + 1 = i\sigma_t''$ for $k_t < i \leq k_{t+1}$. As the domains of σ_t', σ_t'' and σ_u', σ_u'' are disjoint for $t \neq u$, we have $\sigma' = \sigma_0' \ldots \sigma_s'$, $\sigma'' = \sigma_0'' \ldots \sigma_s''$, $\sigma' \sigma''^{-1} = (\sigma_0' \sigma_0''^{-1}) \ldots (\sigma_s' \sigma_s''^{-1})$. Moreover, $i\sigma' \sigma''^{-1} = i\sigma_t' \sigma_t''^{-1}$ implies $\gamma(i, i\sigma' \sigma''^{-1}) = \gamma(i, i\sigma_t' \sigma_t''^{-1}) \in \Lambda_t'$. If $t \in T$ put $\bar{\sigma} = \prod_{t \in T} \sigma_t$; the same argument yields $\bar{\sigma}' = \prod_{t \in T} \sigma_t'$, $\gamma(i, i\sigma' \sigma''^{-1}) = \gamma(i, i\sigma_t' \sigma_t''^{-1}) = \gamma(i, i\bar{\sigma}' \bar{\sigma}''^{-1}) \in (\Gamma_n \mu)'$. Thus, Γ_n' is contained in the right side of (13).

Before proceeding to the converse, we turn to the statement $\Lambda_t' = (\Lambda_t \varphi_t)' \varphi_t'^{-1}$. We can see that φ_t' is an isomorphism, $(\varrho\varphi_t)' = \varrho' \varphi_t'$, $(\varrho\varphi_t)'' = \varrho'' \varphi_t'$ for $\varrho \in P_t$ and, if $k_t < i \leq k_{t+1}$,

$$\gamma(i, i\varrho' \varrho''^{-1}) \varphi_t' = \gamma(i-k_t, i\varrho' \varrho''^{-1} - k_t) = \gamma(i-k_t, (i-k_t)((\varrho' \varrho''^{-1}) \varphi_t') - k_t) =$$
$$= \gamma(i-k_t, (i-k_t)(\varrho\varphi_t)' (\varrho\varphi_t)''^{-1}).$$

Hence $\Lambda_t' \varphi_t' = (\Lambda_t \varphi_t)'$ and $\Lambda_t^* \varphi_t' = (\Lambda_t \varphi_t)^*$. [1]

Now $\Lambda_t \varphi_t$ is f-irreducible and therefore $(\Lambda_t \varphi_t)^*$ is generated by the cyclet $\gamma(j, j\lambda' \lambda''^{-1})$ $(1 \leq j \leq j_t+1, \lambda \in \Lambda_t \varphi_t)$. The same holds for $(\Lambda_t \varphi_t)'$ if $\Lambda_t \varphi_t \nsubseteq \Sigma_{j_t}^{(p)}$ or $\Lambda_t \varphi_t \subseteq A_{j_t}^{(p)}$, i.e. for $t \notin T$. Hence, Λ_t' is also generated by the corresponding cycles $\gamma(i, i\varrho' \varrho''^{-1})$ $(k_t < i \leq k_{t+1}, \varrho \in \Lambda_t)$. By definition of Λ_t, there exists $\sigma \in \Gamma_n$ such that σ_t in (12) equals to ϱ. However then $\gamma(i, i\varrho' \varrho''^{-1}) = \gamma(i, i\sigma' \sigma''^{-1}) \in \Gamma_n'$. Hence $\Lambda_t \subseteq \Gamma_n'$.

Now suppose $t \in T$. Then $\Lambda_t \varphi_t \subseteq \Sigma_{j_t}^{(p)}$, $\Lambda_t \varphi_t \nsubseteq A_{j_t}^{(p)}$ which imply $(\Lambda_t \varphi_t)' = \Sigma_{j_t-1}$, $(\Lambda_t \varphi_t)^* = A_{j_t+1}$. This last gives $\Lambda_t^* \cong A_{j_t+1}$ and so $\Lambda_t^* = P_t' \cap A_n$. As in the fore-

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[1] Remark that this immediately gives $\Gamma_n^* = \prod_{t=0}^{s} \Lambda_t^*$.

going paragraph for Λ_t', now one can verify $\Lambda_t^* \subseteq \Gamma_n'$. Thus $\Gamma_n' \supseteq \prod\limits_{t \in T} \Lambda_t^* \cong \prod\limits_{t \in T} A_{j_t+1}$.
Furthermore, $(\Gamma_n \mu)'$ is generated by the elements of Λ_t^* ($t \in T$) and by those of the form $(\sigma\mu)'$ ($\sigma \in \Gamma_n$). Since $\Lambda_t^* \subseteq A_{n+1}$ and $(\sigma\mu)_t = \sigma_t'$ for $t \in T$, these generators and hence $(\Gamma_n \mu)'$, too, are contained in the right side of (14). On the other hand if for $\tau = \prod\limits_{t \in T} \tau_t$, $\tau_t \in P_t'$ there exists $\sigma \in \Gamma_n$ such that $\tau_t \in A_{n+1} \Leftrightarrow \sigma_t \in A_n$ then $\lambda = (\sigma\mu)'^{-1} \tau \in$
$\in \prod\limits_{t \in T} (P_t' \cap A_{n+1}) = \prod\limits_{t \in T} \Lambda_t^*$ which is obviously contained in the right side of (14); so is $(\sigma\mu)'$ and hence the same holds for $\tau = (\sigma\mu)' \lambda$.

Finally, we have seen $\Lambda_t^* \subseteq \Gamma_n'$ for all t; thus, in order to prove $(\Gamma_n \mu)' \subseteq \Gamma_n'$ it suffices to show $(\sigma\mu)' \in \Gamma_n'$ for $\sigma \in \Gamma_n$. But $\sigma' = (\sigma\mu)' \prod\limits_{t \notin T} \sigma_t'$ and $\prod\limits_{t \notin T} \sigma_t' \in \Gamma_n'$ has been proved earlier. This completes the proof.

It follows from this theorem that the consequence group of an arbitrary group is a direct product of a certain number of permutation groups of type Δ' and of one further group which is an extension of a direct product of alternating groups by an elementary 2-group. The second consequence group is a direct product of symmetric groups.

Suppose $l \geqq 1$ is the maximal number of consecutive integers $1 < k, k+1, \ldots$
$\ldots, k+l-1 < n$ such that $\Gamma_n \subseteq \bigcap\limits_{t=0}^{l-1} \Phi_{k+t}$, $\Gamma_n \nsubseteq \Phi_{k-1}$, $\Gamma_n \nsubseteq \Phi_{k+l}$. Then the $(l+1)$st consequence group is the first one being isomorphic to a symmetric group and if neither 1 nor n is invariant under Γ_n then $\Gamma_n^{(l+1)}$ *is a symmetric group*. This last holds for either Γ_n' or Γ_n'' if Γ_n is f-irreducible.

4. A class of finitely based varieties

The results of this part follow already from the theorem of Putcha and Yaqub cited in the introduction. However, in order to simplify the proof, we shall make use of a result of Perkins, too.

First of all remark that *every variety defined by a set of permutation identities is finitely based.* This follows immediately from the above results.

Combining this fact with the result of Perkins already mentioned, claiming that any uniformly periodic permutative variety is finitely based, we obtain

Theorem 3. *Let* $\mathfrak{S}$ *be a semigroup variety such that in* $\mathfrak{S}$ *hold two (not necessarily different) permutation equations*

$$x_1 \ldots x_n = x_{1\sigma} \ldots x_{n\sigma} \qquad (\sigma \in \Sigma_n, \quad 1\sigma \neq 1),$$

$$x_1 \ldots x_m = x_{1\tau} \ldots x_{m\tau} \qquad (\tau \in \Sigma_m, \quad m\tau \neq m).$$

Then $\mathfrak{S}$ *is finitely based.*

Proof. Put $N=\max (m, n)$; then neither 1 nor N are invariant under Γ_N. Thus, there exists a number l such that the l-th consequence group of Γ_N is Σ_{N+l}. Now there are two cases. Either all identities in $\mathfrak{S}$ are balanced[2]); then the identities of length $\prec N+l$ form a base. Or a non-balanced identity holds in $\mathfrak{S}$; however then holds an identity of the form $x^{k+d}=x^k$, too, and $\mathfrak{S}$ is finitely based in virtue of Perkins's result.

References

[1] M. HALL, *The theory of groups* (New York, 1959).
[2] P. PERKINS, Bases for equational theories of semigroups, *J. Algebra,* **11** (1968), 298—314.
[3] M. S. PUTCHA—A. YAQUB, Semigroups satisfying permutation identities, *Semigroup Forum,* 3 (1971), 68—73.
[4] M. YAMADA, *The structure of separative bands,* Dissertation (Univ. of Utah, 1962).

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[2]) That is, each variable occurs on both sides the same number of times.

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