A new law of the iterated logarithm for multiplicative systems

By P. RÉVÉSZ in Budapest

To Professor B. Sz.-Nagy on his 60th birthday

Introduction

The sequence $\varphi_1, \varphi_2, \ldots$ of random variables on $(X, \mathscr{F}, \mathbf{P})$ is called a multiplicative system (MS) if

$$\int \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_k} = 0 \qquad (i_1 < i_2 < \cdots < i_k;\ k = 1, 2, \ldots),$$

it is called an equinormed strongly multiplicative system (ESMS) if

$$\int \varphi_i = 0, \ \int \varphi_i^2 = 1 \qquad (i = 1, 2, \ldots),$$

$$\int \varphi_{i_1}^{r_1} \varphi_{i_2}^{r_2} \ldots \varphi_{i_k}^{r_k} = \int \varphi_{i_1}^{r_1} \int \varphi_{i_2}^{r_2} \ldots \int \varphi_{i_k}^{r_k} \qquad (i_1 < i_2 < \cdots < i_k;\ k = 1, 2, \ldots),$$

where r_i $(i = 1, 2, \ldots, k)$ can be equal to 1 or 2.

Several theorems state that the properties of a MS resp. ESMS are very similar to those of independent systems.

The best known laws of the iterated logarithm for a MS are the following:

Theorem A. (S. TAKAHASHI[1]) [1].) *Let* $\varphi_1, \varphi_2, \ldots$ *be a uniformly bounded* MS *for which*

(1) $$\int \varphi_i^2 = \int \varphi_i^2 \varphi_j^2 = 1 \qquad (i = 1, 2, \ldots;\ j = 1, 2, \ldots,\ i \neq j).$$

Then

$$\varlimsup_{n\to\infty} \frac{\varphi_1 + \varphi_2 + \cdots + \varphi_n}{\sqrt{2n \log \log n}} \leqq 1 \qquad \textit{a.e.}$$

In fact Takahashi assumed (instead of (1)) that

(2) $$\frac{\varphi_1^2 + \varphi_2^2 + \ldots + \varphi_{[\theta^n]}^2}{[\theta^n]} \to 1 \qquad \text{a.e.}$$

for any $\theta > 1$. (Clearly (1) implies (2).)

[1]) See also [2].

Theorem B. (GAPOSHKIN [3].) *Let* $\varphi_1, \varphi_2, \ldots$ *be a uniformly bounded* ESMS *and let* $a_1, a_2, \ldots$ *be a sequence of real numbers for which*

$$a_n = o\left(\frac{A_n}{\sqrt{\log\log n}}\right) \quad \text{and} \quad A_n \to \infty$$

where $A_n^2 = \sum_{k=1}^{n} a_k^2$. *Then*

$$\overline{\lim_{n\to\infty}} \frac{\sum_{k=1}^{n} a_k \varphi_k}{\sqrt{2A_n^2 \log\log A_n}} \leqq 1 \qquad \text{a.e.}$$

Theorem C. (RÉVÉSZ [4].) *Let* $\varphi_1, \varphi_2, \ldots$ *be a uniformly bounded* MS *for which*

$$\int \varphi_{i_1}^2 \varphi_{i_2}^2 \ldots \varphi_{i_k}^2 = 1 \qquad (i_1 < i_2 < \cdots < i_k;\ k = 1, 2, \ldots).$$

Then

$$\overline{\lim_{n\to\infty}} \frac{\varphi_1 + \varphi_2 + \cdots + \varphi_n}{\sqrt{2n \log\log n}} \geqq 1 \qquad \text{a.e.}$$

In this paper we intend to find a common generalization of Theorems A and B. Our theorem can be formulated as follows:

Theorem 1. *Let* $\varphi_1, \varphi_2, \ldots$ *be a uniformly bounded* MS *and let* $a_1, a_2, \ldots$ *be a sequence of real numbers for which*

$$a_n = o\left(\frac{A_n}{\sqrt{\log\log n}}\right) \quad \text{and} \quad A_n \to \infty, \tag{3}$$

where $A^2(n) = A_n^2 = \sum_{k=1}^{n} a_k^2$. *Further, let* $M_k = M_k(\theta)$, *defined by*

$$A_{M_k - 1}^2 < \theta^k \leqq A_{M_k}^2 \tag{4}$$

and suppose

$$\underline{\lim_{k\to\infty}} \frac{T_{M_k}^2}{A_{M_k}^2} > 0 \qquad \text{a.e.} \tag{5}$$

for any $\theta > 1$, *where* $T^2(n) = T_n^2 = \sum_{k=1}^{n} a_k^2 \varphi_k^2$. *Then*

$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2T_n^2 \log\log A_n}} \leqq 1 \qquad \text{a.e.,}$$

where $S(n) = S_n = \sum_{k=1}^{n} a_k \varphi_k$.

Remark. This theorem is clearly a generalization of Theorem A. The fact that it is also a generalization of Theorem B is shown in the consequence of Lemma 4.

The proof of this Theorem is essentially based on that of TAKAHASHI [1].

§ 1 contains some inequalities. The proof of Theorem 1 is prepared in § 2.

§ 1. Inequalities

Theorem D. (Azuma [5].) *Let $\varphi_1, \varphi_2, \dots, \varphi_n$ be a uniformly bounded* MS ($|\varphi_i| \leqq K_1$, $i = 1, 2, \dots, n$) *and let $a_1, a_2, \dots, a_n$ be a sequence of real numbers, further let λ be a positive number. Then*

$$\int e^{\lambda S} \leqq \exp\left(\frac{\lambda^2 A^2 K_1^2}{2}\right),$$

where $S = \sum_{k=1}^{n} a_k \varphi_k$ *and* $A^2 = \sum_{k=1}^{n} a_k^2$.

We reproduce the proof here because the original one contains a minor miss-print.

Proof. Since e^x is a convex function, for $|x| \leqq 1$ and $a \neq 0$ we have

$$e^{ax} \leqq e^{|a|} \frac{|a| + ax}{2|a|} + e^{-|a|} \frac{|a| - ax}{2|a|} = \operatorname{ch}(|a|) + \frac{ax}{|a|} \operatorname{sh}(|a|).$$

Hence

$$\int e^{\lambda S} = \int \prod_{k=1}^{n} \exp\left(a_k \lambda K_1 \frac{\varphi_k}{K_1}\right) \leqq$$

$$\leqq \int \prod_{k=1}^{n} \left[\operatorname{ch}(|a_k| \lambda K_1) + \frac{a_k}{|a_k|} \frac{\varphi_k}{K_1} \operatorname{sh}(|a_k| \lambda K_1)\right] =$$

$$= \prod_{k=1}^{n} \operatorname{ch}(|a_k| \lambda K_1) = \prod_{k=1}^{n} \sum_{m=0}^{\infty} \frac{(\lambda K_1 |a_k|)^{2m}}{(2m)!} \leqq \prod_{k=1}^{n} \sum_{m=0}^{\infty} \frac{(\lambda K_1 |a_k|)^{2m}}{2^m m!} =$$

$$= \prod_{k=1}^{n} \exp\left(\frac{\lambda^2 K_1^2 a_k^2}{2}\right) = \exp\left(\frac{\lambda^2 K_1^2 A^2}{2}\right),$$

i.e., Theorem D is proved.

Theorem 2. *Let $\varphi_1, \varphi_2, \dots, \varphi_n$ ($|\varphi_i| \leqq K_1$; $i = 1, 2, \dots, n$) be a sequence of uniformly bounded random variables and let $a_1, a_2, \dots, a_n$ be a sequence of real numbers. Then*

$$\exp\left(\lambda S - \frac{\lambda^2 T^2}{2}(1 + 2\lambda K_1 \max_{1 \leqq k \leqq n} |a_k|)\right) \leqq \prod_{k=1}^{n} (1 + \lambda a_k \varphi_k)$$

where

$$S = \sum_{k=1}^{n} a_k \varphi_k, \quad T^2 = \sum_{k=1}^{n} a_k^2 \varphi_k^2, \quad A^2 = \sum_{k=1}^{n} a_k^2,$$

and λ is a positive number for which

$$\lambda K_1 \max_{1 \leqq k \leqq n} |a_k| \leqq \tfrac{1}{2}.$$

Proof. Since

$$e^x \leqq (1+x)\exp\left(\frac{x^2}{2}+|x^3|\right) \quad \text{if} \quad |x| \leqq \tfrac{1}{2}$$

we have

$$\exp(\lambda S) \leqq \prod_{k=1}^{n}(1+\lambda a_k \varphi_k)\exp\left(\frac{\lambda^2 a_k^2 \varphi_k^2}{2}+\lambda^3 |a_k \varphi_k|^3\right) \leqq$$

$$\leqq \exp\left(\frac{\lambda^2 T^2}{2}(1+2\lambda K_1 \max_{1\leqq k\leqq n}|a_k|)\right)\prod_{k=1}^{n}(1+\lambda a_k \varphi_k)$$

which implies our Theorem.

Theorem 3. *Let* $\varphi_1, \varphi_2, \dots, \varphi_n$ *be a uniformly bounded* MS $(|\varphi_i| \leqq K_1;\ i=1,2,\dots,n)$ *and let* $a_1, a_2, \dots, a_n$ *be a sequence of real numbers, further let* y *be a positive number. Then* $\mathbf{P}\{|S| \geqq yK_1 A\sqrt{2}\} \leqq 2e^{-y^2}$, *where* $S=\sum_{k=1}^{n} a_k\varphi_k$ *and* $A^2 = \sum_{k=1}^{n} a_k^2$.

Proof. Set $\lambda=(\sqrt{2}y)/(K_1 A)$. Then by Theorem D we have

$$\int e^{\lambda|S|} \leqq \int e^{\lambda S}+\int e^{-\lambda S} \leqq 2\exp\left(\frac{\lambda^2 A^2 K_1^2}{2}\right),$$

and the Markov inequality gives

$$\mathbf{P}(|S| \geqq yK_1 A\sqrt{2}) = \mathbf{P}(e^{\lambda|S|} \geqq \exp(\lambda y K_1 A\sqrt{2})) \leqq$$

$$\leqq 2\exp\left(\frac{\lambda^2 A^2 K_1^2}{2}-\lambda y K_1 A\sqrt{2}\right) = 2\exp(y^2-2y^2) = 2e^{-y^2},$$

which proves our Theorem 3.

Consequence of Theorem 3. *Let* $\varphi_1, \varphi_2, \dots$ $(|\varphi_i| \leqq K_1;\ i=1,2,\dots)$ *be a sequence of uniformly bounded random variables for which*

$$\int \varphi_{i_1}^2 \varphi_{i_2}^2 \dots \varphi_{i_k}^2 = 1 \qquad (i_1<i_2<\dots<i_k;\ k=1,2,\dots)$$

and let $a_1, a_2, \dots$ *be a sequence of real numbers satisfying condition* (3). *Then*

$$\mathbf{P}\left\{\left|\frac{a_1^2\varphi_1^2+\dots+a_n^2\varphi_n^2}{a_1^2+\dots+a_n^2}-1\right| \geqq \varepsilon\right\} \leqq 2\exp\left(-\frac{\varepsilon^2}{2(K_1^2+1)^2}\,\frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{\sum_{k=1}^{n} a_k^4}\right) \leqq$$

$$\leqq 2\exp\left(-\log\log\left(\sqrt{\sum_{k=1}^{n} a_k^2}\right)\right)$$

for any $\varepsilon>0$ *if* n *is large enough.*

Proof. Clearly $\{\varphi_k^2-1\}$ is a MS. Hence by Theorem 3,

$$\mathbf{P}\left\{\left|\frac{a_1^2\varphi_1^2+\cdots+a_n^2\varphi_n^2}{a_1^2+\cdots+a_n^2}-1\right|\geqq\varepsilon\right\}=\mathbf{P}\left\{\left|\sum_{k=1}^n a_k^2(\varphi_k^2-1)\right|\geqq\varepsilon\sum_{k=1}^n a_k^2\right\}=$$

$$=\mathbf{P}\left\{\left|\sum_{k=1}^n a_k^2(\varphi_k^2-1)\right|\geqq\sqrt{2}\,\frac{\varepsilon\sum_{k=1}^n a_k^2}{\sqrt{2}\,(K_1^2+1)\sqrt{\sum_{k=1}^n a_k^4}}(K_1^2+1)\sqrt{\sum_{k=1}^n a_k^4}\right\}\leqq$$

$$\leqq 2\exp\left(-\frac{\varepsilon^2}{2(K_1^2+1)^2}\,\frac{\left(\sum_{k=1}^n a_k^2\right)^2}{\sum_{k=1}^n a_k^4}\right).$$

Since (3) implies

$$\max_{1\leqq k\leqq n}|a_k|=o\left(\frac{A_n}{\sqrt{\log\log A_n}}\right)\qquad\left(A_n^2=\sum_{k=1}^n a_k^2\right),$$

we have

$$\frac{\left(\sum_{k=1}^n a_k^2\right)^2}{\sum_{k=1}^n a_k^4}\geqq\frac{\left(\sum_{k=1}^n a_k^2\right)^2}{\left(\sum_{k=1}^n a_k^2\right)\left(\max_{1\leqq k\leqq n}|a_k|\right)^2}\geqq\frac{4(K_1^2+1)^2}{\varepsilon^2}\log\log\left(\sqrt{\sum_{k=1}^n a_k^2}\right)$$

f n is large enough, and this proves the consequence.

Theorem 4. *Let $\varphi_1,\varphi_2,\dots,\varphi_n$ be a uniformly bounded* MS $(|\varphi_i|\leqq K_1;$ *$i=1,2,\dots,n$) and let $a_1,a_2,\dots,a_n$ be a sequence of real numbers. Then*

(6) $$\mathbf{P}\left\{\max_{1\leqq m\leqq n}|S(m)|\geqq K_2\sqrt{A^2\log\log A}\right\}\leqq K_3\exp(-2\log\log A)$$

where

$$S(m)=\sum_{k=1}^m a_k\varphi_k,\quad A^2=\sum_{k=1}^n a_k^2.$$

K_2 and K_3 are suitable positive constants.

Before the proof of this theorem we introduce some notations: Let $a_1,a_2,\dots$ be a sequence of real numbers and let

$$I=\{m,m+1,\dots,n\}=[m,n]\qquad(m\leqq n)$$

be the interval of the integers between m and n. Let VI be a partition of I. In its definition we distinguish two cases:

Case 1. There exists ϱ such that $m\leqq\varrho\leqq n$ and $a_\varrho^2\geqq\frac{1}{2}\sum_{i=m}^n a_i^2$.

Case 2. Such an integer does not exist.

In Case 1,

$$\mathrm{V}I = \{\{m, m+1, \ldots, \varrho-1\}, \{\varrho\}, \{\varrho+1, \varrho+2, \ldots, n\}\}.$$

(Of course it can happen that one of these intervals is empty.)

In Case 2,

$$\{\mathrm{V}I = \{m, m+2, \ldots, \tau\}, \{\tau+1, \tau+2, \ldots, n\}\},$$

where τ is defined by

$$\left| \sum_{i=m}^{\tau} a_i^2 - \sum_{i=\tau+1}^{n} a_i^2 \right| = \min.$$

Now let P be a sequence of intervals:

$$P = \{[m_1, n_1], [m_2, n_2], \ldots, [m_s, n_s]\} = \{I_1, I_2, \ldots, I_s\}$$

$$(m_1 \leqq n_1 < m_2 \leqq n_2 < \cdots < m_s \leqq n_s).$$

Then we define UP as the subsequence of P containing those elements (of P) which have more than 1 element (integer).

Finally let

$$\mathrm{V}P = \{\mathrm{V}I_1, \mathrm{V}I_2, \ldots, \mathrm{V}I_s\}.$$

Now construct the sequence $P_0, P_1, \ldots$ as follows:

$$P_0 = \{[1, n]\} \quad \text{and} \quad P_{t+1} = \mathrm{V}UP_t \qquad (t = 0, 1, 2, \ldots).$$

We mention the following two simple properties of the sequence $P_0, P_1, \ldots$.

Property 1. If μ_t is the number of the elements of P_t then $\mu_t \leqq 3.2^{t-1}$ $(t = 1, 2, \ldots)$.

Property 2. If $I_{tj} \in P_t$ then

$$A^2(t, j) \leqq (\tfrac{3}{4})^{t-1} A^2(I) \qquad \left(A^2(t, j) = \sum_{k \in I_{tj}} a_k^2;\ A^2(I) = \sum_{k=1}^{n} a_k^2 \right).$$

Now we can turn to the

Proof of Theorem 4. Clearly we have

$$\max_{1 \leqq m \leqq n} |S(m)| \leqq 2 \sum_{t=0}^{\infty} \max_{1 \leqq j \leqq \mu_t} |S(t, j)|$$

where

$$S(t, j) = \sum_{k \in I_{t,j}} a_k \varphi_k; \qquad \{I_{t1}, I_{t2}, \ldots, I_{t\mu_t}\} = P_t.$$

Set

$$y_t = \sqrt{2 \log\log A + 2t}, \qquad A^2 = \sum_{k=1}^{n} a_k^2,$$

$$F_t = \bigcup_{j=1}^{\mu_t} \{|S(t, j)| \geqq \sqrt{2}\, y_t K_1 A(t, j)\}, \qquad E = \bigcup_{t=0}^{\infty} F_t.$$

Then by Theorem 3 we have

$$\mathbf{P}\left\{|S(t,j)| \geqq \sqrt{2}\, y_t K_1 A(t,j)\right\} \leqq 2e^{-y_t^2} = 2\,\frac{e^{-2\log\log A}}{e^{2t}}$$

hence

$$\mathbf{P}(F_t) \leqq \sum_{j=1}^{\mu_t} 2\,\frac{e^{-2\log\log A}}{e^{2t}} \leqq \frac{3}{2^t}\, e^{-2\log\log A}$$

and

$$\mathbf{P}(E) \leqq \sum_{t=0}^{\infty} \mathbf{P}(F_t) \leqq 6e^{-2\log\log A}. \tag{7}$$

Clearly if $x \notin F_t$ then

$$\max_{1 \leqq j \leqq \mu_t} |S(t,j)| \leqq \sqrt{2}\, K_1 \sqrt{2\log\log A + 2t} \max_{1 \leqq j \leqq \mu_t} A(t,j) \leqq$$

$$\leqq \sqrt{2}\, K_1 \sqrt{2t} \max_{1 \leqq j \leqq \mu_t} A(t,j) + \sqrt{2}\, K_1 \sqrt{2\log\log A} \max_{1 \leqq j \leqq \mu_t} A(t,j) \leqq$$

$$\leqq \sqrt{2}\, K_1 \sqrt{2t} \left(\sqrt{\tfrac{3}{4}}\right)^{t-1} A + \sqrt{2}\, K_1 \sqrt{2\log\log A} \left(\sqrt{\tfrac{3}{4}}\right)^{t-1} A$$

and if $x \notin E$ then

$$\max_{1 \leqq m \leqq n} |S(m)| \leqq \left[4K_1 \sum_{t=0}^{\infty} \sqrt{t} \left(\sqrt{\tfrac{3}{4}}\right)^{t-1} + K_1 \sqrt{\log\log A} \sum_{t=0}^{\infty} \left(\sqrt{\tfrac{3}{4}}\right)^{t-1}\right] A \leqq \tag{8}$$

$$\leqq K_2 \sqrt{A^2 \log\log A}.$$

(7) and (8) imply (6).

§ 2. The proof of Theorem 1

First we prove several lemmas.

Lemma 1. *Under the conditions and notations of Theorem* 1 *we have*

$$\mathbf{P}\left\{\frac{S(M_k)}{\sqrt{2A^2(M_k)\log\log A(M_k)}} \geqq \frac{T^2(M_k)}{2CA^2(M_k)}\left(1 + 2\lambda K_1 \max_{1 \leqq j \leqq M_k} |a_j|\right) + (1+\varepsilon)\frac{C}{2}\right\} =$$

$$= O\left(\frac{1}{k^{1+\varepsilon}}\right)$$

for any $C > 0$ *where*

$$\lambda = \lambda(C) = \sqrt{\frac{2\log\log A(M_k)}{C^2 A^2(M_k)}}.$$

Proof. Set

$$y = (1+\varepsilon)\, C \sqrt{\frac{A^2(M_k)\log\log A(M_k)}{2}}.$$

Since condition (3) implies

$$\max_{1\le j<N} |a_j| = o\left(\frac{A_N}{\sqrt{\log\log A_N}}\right),$$

we have

$$\lambda K_1 \max_{1\le j<M_k} |a_j| = \sqrt{\frac{2\log\log A(M_k)}{C^2 A^2(M_k)}}\, K_1 \max_{1\le j\le M_k} |a_j| \le \tfrac{1}{2}$$

(if k is large enough). Furthermore, Theorem 2 implies

$$\mathbf{P}\left\{\frac{S(M_k)}{\sqrt{2A^2(M_k)\log\log A(M_k)}} \ge \frac{T^2(M_k)}{2A^2(M_k)C}(1+2\lambda K_1 \max_{1\le j\le M_k}|a_j|)+(1+\varepsilon)\frac{C}{2}\right\} =$$

$$= \mathbf{P}\left\{S(M_k) \ge \frac{\lambda}{2} T^2(M_k)(1+2\lambda K_1 \max_{1\le j\le M_k}|a_j|)+y\right\} =$$

$$= \mathbf{P}\left\{\exp\left(\lambda S(M_k) - \frac{\lambda^2}{2} T^2(M_k)(1+2\lambda K_1 \max_{1\le j\le M_k}|a_j|)\right) \ge e^{\lambda y}\right\} \le$$

$$\le e^{-\lambda y} = O\left(\frac{1}{k^{1+\varepsilon}}\right),$$

i.e., Lemma 1 is proved.

Lemma 2. *Under the conditions of Theorem* 1 *for any $\varrho>0$ one can find a set $F(\in\mathcal{F})$, a positive number $\mathcal{K}$ and an integer n_0 such that*

$$\mathbf{P}(F) \le \varrho$$

and $(T_n^2/A_n^2)^{\frac{1}{2}} \ge \mathcal{K}$ hold on $\bar{F}$ if $n \ge n_0$.

Proof. This lemma is a trivial consequence of (5).

Lemma 3. *Define the event $\mathfrak{A}_k$ by*

$$\mathfrak{A}_k = \left\{\frac{S(M_k)}{\sqrt{2T^2(M_k)\log\log A(M_k)}} \ge 1+\delta\right\} \qquad (\delta>0).$$

Then (under the conditions of Theorem 1) only finitely many $\mathfrak{A}_k$ can occur with probability 1.

Proof. By Lemma 1 among the events

$$\mathfrak{B}_k(C) = \left\{\frac{S(M_k)}{\sqrt{2T^2(M_k)\log\log A(M_k)}} \ge \right.$$

$$\left. \ge \frac{1}{2C}\sqrt{\frac{T^2(M_k)}{A^2(M_k)}}(1+2\lambda(C)K_1\max|a_j|)+(1+\varepsilon)\frac{C}{2\sqrt{\frac{T^2(M_k)}{A^2(M_k)}}}\right\}$$

only finitely many will occur. Let now $\{\gamma_k\}$ be a sequence of random variables taking the values $C_1, C_2, \dots, C_R$ ($k=1, 2, \dots$; $C_i>0$; $i=1, 2, \dots, R$). Then among the events $\mathfrak{B}_k(\gamma_k)$ only finitely many will occur (with probability 1) too.

Define a uniform partition of the interval $(\mathscr{K}, K_1)$ (where $\mathscr{K}$ is defined in Lemma 2):

$$C_1 = \mathscr{K} + \frac{K_1 - \mathscr{K}}{R}, \quad C_2 = \mathscr{K} + 2\frac{K_1 - \mathscr{K}}{R}, \dots, C_R = K_1$$

and let

$$\gamma_k = \begin{cases} \mathscr{K} + i\dfrac{K_1 - \mathscr{K}}{R} & \text{if} \quad \mathscr{K} + (i-1)\dfrac{K_1 - \mathscr{K}}{R} \leqq \sqrt{\dfrac{T^2(M_k)}{A^2(M_k)}} \leqq \mathscr{K} + i\dfrac{K_1 - \mathscr{K}}{R}, \\ 0 & \text{if} \quad \sqrt{\dfrac{T^2(M_k)}{A^2(M_k)}} \leqq \mathscr{K}. \end{cases}$$

Then

$$\overline{\mathfrak{B}_k(\gamma_k)} \cap \bar{F} \subset \Bigg\{ \frac{S(M_k)}{\sqrt{2T^2(M_k)\log\log A(M_k)}} \leqq \frac{1}{2\gamma_k}\sqrt{\frac{T^2(M_k)}{A^2(M_k)}}\left(1 + 2\lambda(C_k) \max_{1 \leqq j \leqq M_k} |a_j|\right) +$$

$$+ (1+\varepsilon)\frac{\gamma_k}{2\sqrt{\dfrac{T^2(M_k)}{A^2(M_k)}}} \leqq$$

$$\leqq \tfrac{1}{2}\left(1 + 2\sqrt{\frac{2\log\log A(M_k)}{\mathscr{K}^2 A^2(M_k)}}K_1 \max|a_j|\right) + (1+\varepsilon)\frac{\sqrt{\dfrac{T^2(M_k)}{A^2(M_k)}} + \dfrac{1}{R}}{2\sqrt{\dfrac{T^2(M_k)}{A^2(M_k)}}} \leqq$$

$$\leqq \tfrac{1}{2}(1 + o(1)) + \frac{(1+\varepsilon)}{2}\left(1 + \frac{1}{R\mathscr{K}}\right) \leqq 1 + \delta \Bigg\}$$

if k is large enough and ε and R are chosen in a suitable way. This proves our Lemma 3.

Lemma 4. *Set*

$$F_k = \left\{ \max_{M_k \leqq N < M_{k+1}} |S(N) - S(M_k)| \geqq \varepsilon K_2 \sqrt{A^2(M_k)\log\log A(M_k)} \right\}$$

for any $\varepsilon > 0$. Then (under the conditions of Theorem 1) among the events F_k only finitely many occur with probability 1.

Proof. Since

$$\frac{\sqrt{[A^2(M_{k+1}-1) - A^2(M_k)]\log\log\sqrt{A^2(M_{k+1}-1) - A^2(M_k)}}}{\sqrt{A^2(M_k)\log\log A(M_k)}} \leqq \varepsilon$$

(if k is large enough and θ is chosen near to 1), by Theorem 4 we have

$$\mathbf{P}(F_k) \leqq \mathbf{P}\Big\{ \max_{M_k \leqq N < M_{k+1}} |S(N)-S(M_k)| \geqq$$

$$\geqq K_2 \sqrt{[A^2(M_{k+1}-1)-A^2(M_k)] \log\log \sqrt{A^2(M_{k+1}-1)-A^2(M_k)}} \Big\} \leqq$$

$$\leqq K_3 \exp\left(-2 \log\log \sqrt{A^2(M_{k+1}-1)-A^2(M_k)}\right) = O\left(\frac{1}{k^2}\right)$$

i.e., Lemma 4 is proved.

This lemma, the Consequence of Theorem 3 and the simple relation

$$\sqrt{\sum_{j=1}^{M_k} a_j^4 \log\log \sqrt{\sum_{l=1}^{M_k} a_l^4}} = o\left(\sum_{j=1}^{M_k} a_j^2\right)$$

immediately imply

Consequence of Lemma 4. *Let $\varphi_1, \varphi_2, \dots$ be a sequence of uniformly bounded random variables for which*

$$\int \varphi_{i_1}^2 \varphi_{i_2}^2 \dots \varphi_{i_k}^2 = 1 \qquad (i_1 < i_2 < \dots < i_k;\ k = 1, 2, \dots)$$

and let $a_1, a_2, \dots$ be a sequence of real numbers satisfying condition (3). *Then*

$$\mathbf{P}\left(\frac{T_n^2}{A_n^2} \to 1\right) = 1.$$

Finally, Theorem 1 is a trivial consequence of Lemmas 3 and 4.

References

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