Analytic relations between functional models for contractions

By ION SUCIU in Bucharest (Romania)

To Professor Béla Sz.-Nagy on his 60th anniversary

1. Introduction. The Sz.-Nagy—C. Foiaş functional calculus with bounded analytic functions leads to several results in the study of contractions by means of classical theorems from the analytic function theory.

In this paper, we are going to show how a generalization to functional calculi of two contractions (Theorem 1) of the Harnack inequalities for positive harmonic functions allows us to establish some analytic relations between their Sz.-Nagy—Foiaş functional models (Theorems 2, 3).

We shall use the terminology and notations of [7]. The unitary dilation of the contraction T on the Hilbert space $\mathfrak{H}$ will be denoted by a triplet $[\mathfrak{K}, V, U]$ where $\mathfrak{K}$ is a Hilbert space, V is the isometric embedding of $\mathfrak{H}$ into $\mathfrak{K}$ and U a unitary operator on $\mathfrak{K}$ such that

$$\mathfrak{K} = \bigvee_{n=-\infty}^{\infty} U^n V \mathfrak{H}$$

and

$$T^n = V^* U^n V \qquad (n = 0, 1, 2, \ldots)$$

All notations used in [7] for the geometric structure of the unitary dilation will be rewritten here according to this convention. For example

$$\mathfrak{K}_+ = \bigvee_{n=0}^{\infty} U^n V \mathfrak{H}$$

$$\mathfrak{L} = \overline{(U - VTV^*) V\mathfrak{H}}, \quad \mathfrak{L}^* = \overline{(U - VT^*V^*) V\mathfrak{H}}, \quad \mathfrak{L}_* = \overline{(I - UVT^*V^*) V\mathfrak{H}}.$$

D will stand for the unit disc $\{|z|<1\}$ of the complex plane and X for the unit circle $\{|z|=1\}$. $C(X)$ will denote the C^*-algebra of all continuous complex valued functions on X and A the subalgebra of $C(X)$ containing all functions in $C(X)$

which have analytic extension in D. For $f \in C(X)$ we shall write

$$f(T) = V^* f(U) V.$$

Then $f \to f(T)$ $(f \in C(X))$ is a linear positive map of $C(X)$ into $B(\mathfrak{H})$ the restriction of which to A is an algebra homomorphism of A into $B(\mathfrak{H})$ such that, for any polynomial p in A, $p(T)$ has its usual meaning.

2. Harnack part. Recall that for an integer j the symbol $T^{(j)}$ stands for T^j if $j \geqq 0$ and for T^{*-j} if $j<0$. The main result of this section is:

Theorem 1. ([6]) *Let T_1, T_2 be two contractions on a Hilbert space $\mathfrak{H}$. Let $[\mathfrak{K}^1, V_1, U_1]$, $[\mathfrak{K}^2, V_2, U_2]$ be their unitary dilations and a a number such that $0<a<1$. The following assertions are equivalent:*

(i) *for any polynomial p in A for which* $\operatorname{Re} p \geqq 0$ *we have*

$$a \operatorname{Re} p(T_1) \leqq \operatorname{Re} p(T_2) \leqq 1/a \operatorname{Re} p(T_1);$$

(ii) *for any positive function u in $C(X)$ we have*

$$au(T_1) \leqq u(T_2) \leqq 1/a\ u(T_1);$$

(iii) *for any positive integer n, any positive $n \times n$-matrix (u_{ij}) over $C(X)$ and any finite system $h_1, \dots, h_n$ of elements in $\mathfrak{H}$ we have*

$$a \sum_{i,j} (u_{ij}(T_1) h_j, h_i) \leqq \sum_{i,j} (u_{ij}(T_2) h_j, h_i) \leqq 1/a \sum_{i,j} (u_{ij}(T_1) h_j, h_i);$$

(iv) *for any positive integer n and any finite system $h_1, \dots, h_n$ of elements in $\mathfrak{H}$ we have*

$$a \sum_{i,j} (T^{(j-i)} h_j, h_i) \leqq \sum_{i,j} (T_2^{(j-i)} h_j, h_i) \leqq 1/a \sum_{i,j} (T_1^{(j-i)} h_j, h_i);$$

(v) *there exists a linear boundedly invertible operator S from $\mathfrak{K}^2$ onto $\mathfrak{K}^1$, such that $\|S\| \leqq 1/\sqrt{a}$ and*

$$SV_2 = V_1, \quad SU_2 = U_1 S.$$

Proof. The implication (i)⇒(ii) follows from the fact that the real parts of the polynomials in A are uniformly dense in the set of real functions in $C(X)$.

The implication (ii)⇒(iii) comes from the Naĭmark dilation theorem as follows: according to (ii) $f \to f(T_2) - af(T_1)$, $(f \in C(X))$, is a positive linear map of $C(X)$ in $B(\mathfrak{H})$. Let $[\mathfrak{K}, V, \pi)$ be the spectral dilation of this map. Thus $\mathfrak{K}$ is a Hilbert space, V is a bounded operator from $\mathfrak{H}$ into $\mathfrak{K}$ and π a representation of $C(X)$ in $B(\mathfrak{K})$ such that

$$f(T_2) - af(T_1) = V^* \pi(f) V \qquad (f \in C(X))$$

(ese for example [1], [5]). Let $(u_{ij})=(g_{ij})^*(g_{ij})$ be a positive $n\times n$: matrix over $C(X)$ and $h_1, \dots, h_n \in \mathfrak{H}$. We have

$$\sum_{i,j}(u_{ij}(T_2)-au_{ij}(T_1)h_j, h_i) = \sum_{i,j}(V^*\pi(u_{ij})Vh_j, Vh_i) =$$

$$= \sum_{i,j}(V^*\pi(\sum_k \bar{g}_{ki}g_{kj})Vh_j, h_i) = \sum_k \sum_{ij}(\pi(g_{ki})^*\pi(g_k)Vh_j, Vh_i) =$$

$$= \sum_k \| \sum_j \pi(g_{kj})Vh_j\|^2 \geqq 0.$$

One obtains the second inequality in (iii) by simmetry.

Taking (iii) with $(u_{ij})=(g_{ij})^*(g_{ij})$, where $g_{1j}(z)=z^j$, $j=1, 2, \dots, n$ and $g_{ij}(z)=0$ for $i\geqq 2$ we obtain (iv).

Let us prove the implication (iv)$\Rightarrow$(v). For any positive integer n and $h_1, \dots, h_n \in H$ we have

$$a\|\sum_j U_1^j V_1 h_j\|^2 = a\sum_{i,j}(V_1^* U_1^{j-i} V_1 h_j, h_i) = a\sum_{ij}(T_1^{(j-i)}h_j, h_i) \leqq$$

$$\leqq \sum_{ij}(T_2^{(j-i)}h_j, h_i) = \sum_{ij}(V_2^* U_2^{j-i} V_2 h_j, h_i) = \|\sum_j U_2^j V_2 h_j\|^2.$$

Thus there exists a bounded operator S from $\mathfrak{K}^2$ into $\mathfrak{K}^1$ such that $\|S\|\leqq 1/\sqrt{a}$ and

$$S\sum_{j=1} U_2^j V_2 h_j = \sum_{j=1} U_1^j V_1 h_j.$$

The second inequality in (iv) shows that S^{-1} exists and $\|S^{-1}\|\leqq 1/\sqrt{a}$. It is clear that

$$SV_2=V_1, \quad SU_2=U_1 S.$$

Since the implication (ii)$\Rightarrow$(i) is obvious, it remains to prove the implication (v)$\Rightarrow$(ii). To do this, let $K=a(S^{-1})^* S^{-1}$. Then $0\leqq K\leqq I$ and it is easy to see that $Kf(U_1)=f(U_1)K$ for any $f\in C(X)$.

Moreover

$$af(T_2) = aV_2^* f(U_2)V_2 = V_2^* aS^{-1}f(U_1)SV_2 =$$

$$= V_1^* a(S^{-1})^* S^{-1}f(U_1)V_1 = V_1^* Kf(U_1)V_1.$$

Let Z be the positive square root of $I-K$. Then Z commutes with $f(U_1)$ for any $f\in C(X)$. Hence for all positive u in $C(X)$ and h in H we have

$$((u(T_1)-au(T_2))h, h) = ((V_1^* u(U_1)V_1 - V_1^* Ku(U_1)V_1)h, h) =$$

$$= (V_1^*(I-K)u(U_1)V_1 h, h) = (V_1^* Z^2 u(U_1)V_1 h, h) = (u(U_1)ZV_1 h, ZV_1 h) \geqq 0.$$

Hence

$$au(T_2) \leqq u(T_1).$$

The second inequality in (ii) is obtained again by symmetry.

The proof of the theorem is complete.

The inequalities contained in Theorem 1 generalize the Harnack inequalities for positive harmonic functions.

We say that T_1 ant T_2 are *Harnack equivalent* if they satisfy one of the (equivalent) assertions of Theorem 1. (Note that T_1 is always Harnack equivalent with $T_2=T_1$). This equivalence relation determines on the set of all contractions on $\mathfrak{H}$ equivalence classes. Such a class will be called a *Harnack part*. The concept is analogous to that of Gleason parts of the complex homomorphisms of a function algebra (see for example [2]).

Corollary 1. *Two contractions T_1, T_2 are Harnack equivalent if and only if T_1^*, T_2^* are.*

Corollary 2. *If T_1 and T_2 are Harnack equivalent then U_1 and U_2 are unitary equivalent.*

Proof. Using standard arguments we can show that if $S=|S|U$ is the polar decomposition of S then the fact that S has a bounded inverse implies that U is a unitary operator from $\mathfrak{K}^2$ onto $\mathfrak{K}^1$ and $UU_2=U_1U$.

Note that, in general, $UV_2\neq V_1$, thus the two unitary dilations do not coincide.

Corollary 3. *If T is an isometric operator on $\mathfrak{H}$ then the Harnack part containing T reduces to $\{T\}$.*

Proof. Suppose that T_1 is in the same Harnack part as $T_2=T$ and let $[\mathfrak{K}^1, V_1, U_1]$, $[\mathfrak{K}^2, V_2, U_2]$ be the unitary dilations of T_1, T_2, respectively. Let S be the operator defined in Theorem 1. Since T_2 is an isometry we have $V_2T_2=U_2V_2$. Therefore

$$V_1T_2 = SV_2T_2 = SU_2V_2 = U_1V_1$$

Hence

$$T_2 = V_1^*V_1T_2 = V_1^*U_1V_1 = T_1.$$

Corollary 4. *Let T_1, T_2 be in the same Harnack part. Then T_1 and T_2 have the same unitary part. In particular, if T_1 is completely non unitary then so is T_2.*

Proof. The maximal subspaces of $\mathfrak{H}$ which reduce T_i $(i=1, 2)$, to unitary operators are

$$\mathfrak{H}_i = \{h\in\mathfrak{H}: U_iV_ih\in V_iH, \qquad n=0, \pm1, \pm2 \ldots\}.$$

For $h\in\mathfrak{H}_1$ and $n=0, \pm1, \pm2, \ldots$ we have

$$U_2^nV_2h = S^{-1}U_1^nSV_2h = S^{-1}U_1^nV_1h\in V_2\mathfrak{H}.$$

Thus $\mathfrak{H}_1 \subset \mathfrak{H}_2$ and by symmetry $\mathfrak{H}_1 = \mathfrak{H}_2$. Moreover, for $h \in \mathfrak{H}_1 = \mathfrak{H}_2$ we have

$$V_2 T_1 h = S^{-1} V_1 T_1 h = S^{-1} U_1 V_1 h = U_2 S^{-1} V_1 h = U_2 V_2 h = V_2 T_2 h.$$

Thus $T_1 h = T_2 h$.

In [3] C. Foiaş proves that the set $B_0 = \{T \in B(\mathfrak{H}), \|T\| < 1\}$ forms a Harnack part, the Harnack part of the contraction 0. Using this result and Corollary 4 one can also prove (see [3]) that there exist Harnack parts different from B_0 and which contain more than one element.

3. Analyticity of the operator S. Suppose that T_1, T_2 are in the same Harnack part and let S be the operator defined in Theorem 1. Since $SV_2 = V_1$, $SU_2 = U_1 S$, $S_2^* U = U_1^* S$, we have

$$S\mathfrak{K}_+^2 = S \bigvee_{n=0}^{\infty} U_2^n V_2 \mathfrak{H} = \bigvee_{n=0}^{\infty} U_1^n S V_2 \mathfrak{H} = \bigvee_{n=0}^{\infty} U_1^n V_1 \mathfrak{H} = \mathfrak{K}_+^1,$$

$$S\mathfrak{K}_-^2 = S \bigvee_{n=0}^{\infty} U_2^* V_2 \mathfrak{H} = \bigvee_{n=0}^{\infty} U_1^* S V_2 \mathfrak{H} = \bigvee_{n=0}^{\infty} U_1^* V_1 \mathfrak{H} = \mathfrak{K}_-^1.$$

Thus

(3.1) $$S\mathfrak{K}_+^2 = \mathfrak{K}_+^1, \quad S\mathfrak{K}_-^2 = \mathfrak{K}_-^1$$

From (3.1) it follows that

$$S^* M_+(\mathfrak{L}^1) = S^*(\mathfrak{K}^1 \ominus \mathfrak{K}_-^1) \subset \mathfrak{K}^2 \ominus \mathfrak{K}_-^2 = M_+(\mathfrak{L}^2).$$

Hence

(3.2) $$S^* M_+(\mathfrak{L}^1) = M_+(\mathfrak{L}^2).$$

Since $S^* U_1^* = U_2^* S$ (3.2) implies

(3.3) $$S^* M(\mathfrak{L}^1) = M(\mathfrak{L}^2).$$

On the other hand

$$S\mathfrak{R}^2 = S \bigcap_{n=0}^{\infty} U_2^n \mathfrak{K}_+^2 = \bigcap_{n=0}^{\infty} U_1^n K_+^1 = \mathfrak{R}^1.$$

Thus

(3.4) $$S\mathfrak{R}^2 = \mathfrak{R}^1.$$

If $h \in \mathfrak{H}$, then $P_{\mathfrak{R}^2} V_2 h = \lim U_2^n T_2^{*n} h$. Thus $SP_{\mathfrak{R}^2} V_2 h = \lim SU_2^n T_2^{*n} h =$ $= \lim U_1^n V_1 T_2^{*n} h$. But $\|U_1^n V_1 T_2^{*n} h\| = \|V_1 T_2^{*n} h\| = \|V_2 T_2^{*n} h\| = \|U_2^n V_2 T_2^{*n} h\|$. Thus $\|SP_{\mathfrak{R}^2} V_2 T_2^* h\| = \lim \|U_1^n V T_2^{*n} h\| = \lim \|U_2^n V_2 T_2^{*n} h\| = \|P_{\mathfrak{R}^2} V_2 h\|$. Which together prove

(3.5) $$\|SP_{\mathfrak{R}^2} V_2 h\| = \|P_{\mathfrak{R}^2} V_2 h\|, \qquad (h \in H).$$

Put

$$\mathfrak{M}^2 = \overline{P_{\mathfrak{R}^2} V_2 \mathfrak{H}}, \quad \mathfrak{M}^1 = S\mathfrak{M}^2.$$

Since

$$U_2^* P_{\mathfrak{R}^2} V_2 h = P_{\mathfrak{R}^2} V_2 T_2^* h \qquad (h \in \mathfrak{H}).$$

it follows that $U_2^*\mathfrak{M}^2 \subset \mathfrak{M}^2$ and $U_1^*\mathfrak{M}_1 = U_1^* S\mathfrak{M}^2 = SU_2^*\mathfrak{M}^2 \subset S\mathfrak{M}^2 = \mathfrak{M}^1$. Set

$$T_1' = U_1^*|\mathfrak{M}^1, \quad T_2' = U_2^*|\mathfrak{M}^2, \quad S' = S|\mathfrak{M}^2.$$

According to (3. 5), S' is a unitary operator from $\mathfrak{M}^2$ onto $\mathfrak{M}^1$ and

$$S'T_2' = T_1'S'.$$

It is easy to verify that $U_2^*|\mathfrak{R}^2$ and consequently $U_1^*|\mathfrak{R}^1$ are minimal unitary dilations of T_2' and T_1', respectively. Since $SU_2^* = U_1^* S$, and S extends S', by using standard arguments we can conclude that $S|\mathfrak{R}^2$ is a unitary operator from $\mathfrak{R}^2$ onto $\mathfrak{R}^1$.

From (3. 1) it follows that the operator $S_+ = S|\mathfrak{K}_2^+$ from $\mathfrak{K}_2^+$ onto $\mathfrak{K}_1^+$ has a bounded inverse. Since

$$\mathfrak{L}^i = \mathfrak{K}_+^i \ominus U_i \mathfrak{K}_+^i$$

for any $l \in \mathfrak{L}_*^1$ and $k \in \mathfrak{K}_2^+$ we have

$$(S_+^* l, U_2 k) = (l, S_+ U_2 k) = (l, SU_2 k) = (l, U_1 Sk) = 0.$$

Thus $S_+^* \mathfrak{L}_*^1 \subset \mathfrak{L}_+^2$ and by symmetry we obtain

(3. 6) $$S_+^* \mathfrak{L}_*^1 = \mathfrak{L}_*^2.$$

So we have proved the following

Theorem 2. *Let T_1, T_2 be two Harnack equivalent contractions on $\mathfrak{H}$ and let S be the operator defined in Theorem* 1. *Then*

(i) $S^* M(\mathfrak{L}^1) = M(L^2), \quad S^* M_+(\mathfrak{L}^1) = M_+(\mathfrak{L}^2)$;

(ii) *S_+ is a bounded operator from $M_+(\mathfrak{L}_*^2) \oplus \mathfrak{R}^2$ onto $M_+(\mathfrak{L}_*^1) \oplus \mathfrak{R}^1$ which has bounded inverse and*

$$S_+^* \mathfrak{L}_*^1 = \mathfrak{L}_*^2;$$

(iii) *$S|\mathfrak{R}^2$ is a unitary operator from $\mathfrak{R}^2$ onto $\mathfrak{R}^1$.*

From assertions (i) and (ii) of Theorem 2 it follows that $\mathfrak{K}^1 = M(\mathfrak{L}^1)$ $(\mathfrak{K}^1 = M(\mathfrak{L}_*^1))$ if and only if $\mathfrak{K}^2 = M(\mathfrak{L}^2)$ $(\mathfrak{K}^1 = M(\mathfrak{L}_*^2))$. In virtue of Theorem 1. 2, ch. II in [7] we obtain

Corollary 5. *If T_1 and T_2 are Harnack equivalent then T_1 is of class $C_{\cdot 0}$ $(C_{0\cdot}, C_{00})$ if and only if T_2 has this property.*

From assertion (ii) of Theorem 2 and Corollary 1 we conclude

Corollary 6. *If T_1 and T_2 are Harnack equivalent then they have the same defect indices.*

Suppose now that $\mathfrak{H}$ is separable. Taking the Fourier representations of the bilateral shift involved, Theorem 2 allows us to say (according to Lemma 3. 1 Ch.

V in [7]) that in these representations $S^*|M(\mathfrak{L}^1)$ is a bounded analytic function $\{\mathfrak{L}^1, \mathfrak{L}^2, S^*(\lambda)\}$. In the $C_{\cdot 0}$ case S is a bounded analytic function too, namely $\{\mathfrak{L}^2_*, \mathfrak{L}^1_*, S(\lambda)\}$.

In this last case we can establish an analytic relation between characteristic functions as follows.

Theorem 3. *Let T_1 and T_2 be two Harnack equivalent contractions on H. Suppose T_1 (and consequently T_2) belongs to the class $C_{\cdot 0}$. Let $\{\mathfrak{L}^1, \mathfrak{L}^1_*, \theta_1(\lambda)\}$, $\{\mathfrak{L}^2, \mathfrak{L}^2_*, \theta_2(\lambda)\}$ be the characteristic functions of T_1, T_2 respectively. Then there exist bounded, boundedly invertible, analytic functions $\{\mathfrak{L}^2_*, \mathfrak{L}^1_*, S(\lambda)\}$ and $\{\mathfrak{L}^1, \mathfrak{L}^2, \Sigma(\lambda)\}$ such that we have*

$$S(e^{it})^* \theta_1(e^{it}) = \theta_2(e^{it}) \sum(e^{it}) \quad a.e.$$

Proof. Let $\{\mathfrak{L}^2_*, \mathfrak{L}^1_*, S(\lambda)\}$ be the bounded analytic function constructed above. From Theorem 2 it follows that

$$S(e^{it})^* \theta_1(e^{it}) H^2(\mathfrak{L}^1) \subset \theta_2(e^{it}) H^2(\mathfrak{L}^2).$$

Thus we can define the operator Σ by

$$S(e^{it})^* \theta_1(e^{it}) u(t) = \theta_2(e^{it}) (\sum u)(t) \quad (u \in H^2(\mathfrak{L}^1)).$$

It is easy to verify that the operator Σ commutes with the multiplication with e^{it}. It results that Σ arises as multiplication operator from a bounded analytic function $\{\mathfrak{L}^1, \mathfrak{L}^2, \Sigma(\lambda)\}$. The fact that these functions are boundedly invertible results directly from Lemma 3. 2 ch. V in [7] and Theorem 2 above.

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