On the convergence of Hermite—Fejér interpolation based on the roots of the Legendre polynomials

By J. SZABADOS in Budapest*)

To Professor B. Sz.-Nagy on his sixtieth birthday

Let $f(x)$ be an arbitrary continuous function in the interval $[-1, 1]$. If $1>x_1>>x_2>\cdots>x_n>-1$ are the roots of the Legendre polynomial $P_n(x)$ of degree n then the so-called Hermite—Fejér interpolating polynomials

$$H_n(f, x) = \sum_{k=1}^{n} \frac{1-2xx_k+x_k^2}{1-x_k^2}\left(\frac{P_n(x)}{P_n'(x_k)(x-x_k)}\right)^2$$

of degree $\leqq 2n-1$ satisfy

$$H_n(f, x_k) = f(x_k), \quad H_n'(f, x_k) = 0 \qquad (k = 1, \dots, n).$$

It is well known (see FEJÉR [1]) that

$$\lim_{n\to\infty} H_n(f, x) = f(x) \qquad (|x| < 1)$$

for all continuous $f(x)$, and the convergence is uniform in each closed subinterval of $(-1, 1)$. Our first result improves this statement by giving an estimate for the rate of convergence. In what follows, $\omega_f(t)$ will denote the modulus of continuity of $f(x)$.

Theorem 1. *Let $f(x)$ be a continuous function in $[-1, 1]$ then*

$$|f(x)-H_n(f, x)| =$$
$$= \max\left(\left|f(1)-\tfrac{1}{2}\int_{-1}^{1} f(x)\,dx\right|, \left|f(-1)-\tfrac{1}{2}\int_{-1}^{1} f(x)\,dx\right|\right)\cdot O\left(\frac{1}{n\sqrt{1-x^2}}\right)+$$
$$+O\left(\omega_f\left(\frac{\log n}{n}\right)\right) \qquad (|x| < 1).$$

*) Research supported by NSF under Grant GP-9493 while the author visited the Ohio State University, Columbus, Ohio.

Proof. Let $n \geqq 3$,

$$m = \left[\frac{n}{\log n}\right], \tag{1}$$

and $q_m(x)$ be the best approximating polynomial of degree $\leqq m$ to $f(x)$ in $[-1, 1]$. Then by the Jackson theorem

$$\|f(x) - q_m(x)\| = O\left(\omega_f\left(\frac{1}{m}\right)\right) \tag{2}$$

($\|.\|$ means the maximum-norm of the corresponding function in $[-1, 1]$). By the linearity and positivity of the operator H_n we have

$$|f(x) - H_n(f, x)| \leqq |f(x) - q_m(x)| + |q_m(x) - H_n(q_m, x)| + \tag{3}$$

$$+ |H_n(q_m - f, x)| = O\left(\omega_f\left(\frac{1}{m}\right)\right) + |q_m(x) - H_n(q_m, x)|.$$

Assume first that $0 \leqq x < 1$. By $m \leqq 2n-1$ (see (1)) we obtain (cf. e.g. SZEGŐ [2], (14. 1. 9))

$$q_m(x) - H_n(q_m, x) = \sum_{k=1}^{n} q_m'(x_k) \frac{P_n(x)^2}{P_n'(x_k)^2 (x - x_k)} = \tag{4}$$

$$= \tfrac{1}{2} P_n(x)^2 \sum_{k=1}^{n} \frac{2(1+x_k) q_m'(x_k)}{P_n'(x_k)^2 (1 - x_k^2)} + P_n(x)^2 (1-x) \sum_{k=1}^{n} \frac{q_m'(x_k)}{P_n'(x_k)^2 (1 - x_k)(x - x_k)}.$$

Here the first sum is the Gauss—Jacobi quadrature for the polynomial $(1+x)q_m'(x)$ of degree $\leqq m \leqq 2n-1$, thus it is equal to

$$\int_{-1}^{1} (1+x) q_m'(x)\, dx = 2q_m(1) - \int_{-1}^{1} q_m(x)\, dx,$$

which, in turn, tends to $2f(1) - \int_{-1}^{1} f(x)\, dx$ as n (and by (1) m) tend to infinity (see (2)). Therefore, by the inequality

$$P_n(x)^2 \leqq \frac{1}{n\sqrt{1-x^2}} \qquad (|x| < 1) \tag{5}$$

(cf. SZEGŐ [2], Theorem 7. 3. 3), we get from (4)

$$q_m(x) - H_n(q_m, x) = \left| f(1) - \tfrac{1}{2} \int_{-1}^{1} f(x)\, dx \right| O\left(\frac{1}{n\sqrt{1-x^2}}\right) + \tag{6}$$

$$+ O\left(\frac{(1-x^2)^{3/4}}{\sqrt{n}} \sum_{k=1}^{n} \frac{|q_m'(x_k)|}{P_n'(x_k)^2 (1-x_k)} \left|\frac{P_n(x)}{x - x_k}\right|\right) \qquad (0 \leqq x < 1).$$

Using a theorem of S. B. STEČKIN [3] which states that for an arbitrary polynomial $q_m(x)$ of degree $\leqq m$

$$|q_m'(x)| = O\left(\frac{m}{\sqrt{1-x^2}}\right)\cdot \omega_{q_m}\left(\frac{1}{m}\right) \qquad (|x|<1)$$

holds, we get by (2) and $\omega_g(t) \leqq 2\|g\|$ that

$$(7)\qquad |q_m'(x)| = O\left(\frac{m}{\sqrt{1-x^2}}\right)\left[\omega_f\left(\frac{1}{m}\right)+\omega_{q_m-f}\left(\frac{1}{m}\right)\right] = O\left(\frac{m\omega_f\left(\frac{1}{m}\right)}{\sqrt{1-x^2}}\right) \qquad (|x|<1).$$

We also need the following estimates in connection with the Legendre polynomials:

$$(8)\qquad \frac{2k-1}{2n+1}\pi \leqq \theta_k = \arccos x_k \leqq \frac{2k}{2n+1} \qquad (k=1, \dots, n)$$

(SZEGŐ [2], Theorem 6. 21. 2), and

$$(9)\qquad P_n'(x_k) \sim \begin{cases} \theta_k^{-3/2}\sqrt{n} \sim k^{-3/2}n^2 & (1 \leqq k \leqq n/2), \\ (\pi-\theta_k)^{-3/2}\sqrt{n} \sim (n-k)^{-3/2}n^2 & (n/2 \leqq k \leqq n). \end{cases}$$

(SZEGŐ [2], (8. 9. 7)). If we denote $x = \cos\theta \geqq 0$ and $|x-x_j| = \min\limits_{1\leqq k\leqq n} |x-x_k|$ then (5) and (8) imply

$$(10)\qquad \left|\frac{P_n(x)}{x-x_k}\right| = \begin{cases} O(|P_n'(x_j)|) & \text{if } k=j, \\ O\left(\dfrac{n^2}{\sqrt{j}\,|j^2-k^2|}\right) & \text{if } 1<j+k\leqq n,\ k\neq j, \\ O\left(\dfrac{n^2}{\sqrt{j}\,|j-k|(2n-j-k)}\right) & \text{if } n<j+k\leqq 2n. \end{cases}$$

Collecting our estimates (7)—(10) we obtain by (1)

$$\frac{(1-x^2)^{3/4}}{n^{1/2}}\sum_{k=1}^{n}\frac{|q_m'(x_k)|}{P_n'(x_k)^2(1-x_k)}\left|\frac{P_n(x)}{x-x_k}\right| = O\left(\frac{j^{3/2}m}{n^2}\right)\omega_f\left(\frac{1}{m}\right)\left[\frac{1}{j^{-3/2}n^2j^3n^{-3}}+\right.$$

$$+\left(\frac{n^2}{\sqrt{j}}\sum_{\substack{1<j+k\leqq n\\ k\neq j}}\frac{1}{k^{-3}n^4k^3n^3|j^2-k^2|}+\right.$$

$$\left.\left.+\sum_{n<j+k\leqq\frac{3}{2}n}\frac{1}{(n-k)^{-3}n^4(n-k)^3n^{-3}|j-k|(2n-j-k)}\right)\right] =$$

$$= O\left(\frac{1}{\log n}\right)\omega_f\left(\frac{\log n}{n}\right)\left(1+\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{|j-k|}\right) = O\left(\omega_f\left(\frac{\log n}{n}\right)\right) \qquad (0\leqq x<1).$$

This together with (3) and (6) means that

$$|f(x)-H_n(f,x)| = \left|f(1)-\tfrac{1}{2}\int_{-1}^{1} f(x)\,dx\right| O\left(\frac{1}{n\sqrt{1-x^2}}\right)+O\left(\omega_f\left(\frac{\log n}{n}\right)\right)$$
$$(0 \leqq x < 1).$$

A similar estimate holds for $-1<x<0$ and the proof of Theorem 1 is complete.

As for the endpoints ± 1, FEJÉR [1] proved that

$$\lim_{n\to\infty} H_n(f, \pm 1) = \tfrac{1}{2} \int_{-1}^{1} f(x)\,dx. \tag{11}$$

G. FREUD [4] raised the question (in a much more general form) whether the *necessary* condition

$$f(\pm 1) = \tfrac{1}{2} \int_{-1}^{1} f(x)\,dx \tag{12}$$

for

$$\lim_{n\to\infty} \|f(x)-H_n(f,x)\| = 0, \tag{13}$$

obtained from (11), is *sufficient* as well. Recently, A. SCHÖNHAGE [5] has given an answer in the affirmative by proving that (12) implies (13). The following result (which is an easy corollary to our Theorem 1) is an improvement of the Schönhage theorem (namely, it contains an estimate for the rate of convergence).

Theorem 2. *Let $f(x)$ be a continuous function in* $[-1, 1]$ *for which* (12) *holds. Then*

$$\|f(x)-H_n(f,x)\| = O\left(\omega_f\left(\frac{\log n}{n}\right)\right).$$

References

[1] L. FEJÉR, Über Interpolation, *Göttinger Nachrichten* (1916), 66—91.

[2] G. SZEGŐ, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ., Vol. XXIII, 1959.

[3] S. B. STEČKIN, Generalization of some inequalities of S. N. Bernstein, *DAN SSSR*, **60/9** (1948), 1511—1514 (Russian).

[4] G. FREUD, On Hermite—Fejér interpolation sequences, *Acta Math. Acad. Sci. Hung.*, **23** (1972), 175—178.

[5] A. SCHÖNHAGE, Zur Konvergenz der Stufenpolynome über den Nullstellen der Legendre Polynome, *Proceedings of the Conference on Abstract Spaces and Approximation*, held in Oberwolfach, 1971, Birkhäuser Verlag (to appear).

(Received April 1, 1972)