Compactness, metrizability, and Baire isomorphism

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To Béla Sz.-Nagy on his sixtieth birthday

Any comparative study of compact spaces naturally begins with a suitable characterization of compact metric spaces. A topological characterization of this type is given in the present paper. The characterization is based on a result concerning Baire isomorphisms between two spaces. Such isomorphisms between compact metric spaces have been studied in some detail by the authors ([3], [4], [5], [6]).

All spaces considered are separated. Given such a space $\mathbf{X}$, a set $\mathbf{M} \subset \mathbf{X}$ is the zeroset of a continuous real-valued function f in case $\mathbf{M} = \{x : x \in \mathbf{X}, f(x) = 0\}$. The family of Baire sets on $\mathbf{X}$ is the smallest family which contains all zerosets of continuous functions and which is closed under complementation, denumerable union, and denumerable intersection. The family of Baire functions is the smallest family of bounded real-valued functions which contains all bounded continuous real-valued functions and which is closed under the operation of the taking of pointwise limits of sequences of functions. A Baire isomorphism $\Phi : \mathbf{X} \to \mathbf{Y}$ is a bijection such that for any Baire set $\mathbf{M} \subset \mathbf{X}$, $\Phi(\mathbf{M})$ is a Baire set in $\mathbf{Y}$; and such that for any Baire set $\mathbf{N} \subset \mathbf{Y}$, $\Phi^{-1}(\mathbf{N})$ is a Baire set in $\mathbf{X}$. A Baire function is Baire measurable and any Baire measurable function is a Baire function.

A family of real-valued functions defined on $\mathbf{X}$ is said to separate points in case for each $x_1, x_2 \in \mathbf{X}$, $x_1 \neq x_2$, there exists a function f of the family such that $f(x_1) \neq f(x_2)$. Although the role of separation of points for families of continuous functions is universally known, results for separation by noncontinuous functions are less common. Our first theorem concerns such a situation.

Theorem A. *Let $\mathbf{X}$ be a compact space. Then $\mathbf{X}$ is metrizable if and only if there exists a denumerable set of Baire functions $\{\varphi_n\}$ which separates the points of $\mathbf{X}$.*

Proof. If $\mathbf{X}$ is compact and metrizable then there exists a denumerable set of continuous functions $\{\varphi_n\}$, hence, a fortiori, Baire functions, which separate points.

Now let $\{\varphi_n\}$ be a denumerable family of Baire functions which separates points.

Consider one of the functions of the family. Call it φ. We shall say that a set $\mathscr{F}$ of continuous functions is an *ancestral family* for φ in case the smallest family of Baire functions containing $\mathscr{F}$ and closed under the process of the taking of pointwise limits of sequences of functions contains φ. Thus, one ancestral family for φ is the family of all continuous real-valued functions on $\mathbf{X}$. In general, there are many other ancestral families for φ.

It may easily be seen that if φ is a Baire function, there exists at least one denumerable ancestral family for φ. This can be proved by transfinite induction on the order α of the Baire function. If $\alpha=0$, then φ is continuous and $\{\varphi\}$ is an ancestral family for φ. Suppose now that φ is of Baire class α and that the result is valid for every Baire function of order $\beta<\alpha$. Suppose $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$, where φ_n is a Baire function of class $\beta_n<\alpha$. If $\mathscr{G}_n$ is a denumerable ancestral family for φ_n, then $\mathscr{G} = \bigcup_{n=1}^{\infty} \mathscr{G}_n$ is a denumerable ancestral family for φ.

Going back to the proof of the theorem, let $\mathscr{F}_n$ be a denumerable ancestral family of continuous functions for φ_n. Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$. Then $\mathscr{F}$ is a denumerable family of continuous functions. We write $\mathscr{F}=\{f_1, f_2, \ldots\}$.

It is clear that the functions of $\mathscr{F}$ separate points. Indeed, if for $x_1\neq x_2$, $f(x_1)= =f(x_2)$ for each function in $\mathscr{F}$, then for any Baire function φ generated from $\mathscr{F}$, one has $\varphi(x_1)=\varphi(x_2)$. The hypothesis of Theorem A thus implies that $\mathscr{F}$ separates points. Now, let $\Xi:\mathbf{X}\to\mathbf{R}^\omega$ be the map of $\mathbf{X}$ into the denumerable product of $\mathbf{R}$, given by $\Xi(x)=(f_1(x), f_2(x), \ldots)$. Then Ξ is a continuous bijection of $\mathbf{X}$ onto the subset $\mathbf{Y}=\Xi(\mathbf{X})$ of $\mathbf{R}^\omega$. Since $\mathbf{X}$ is compact, so is $\mathbf{Y}$. Since Ξ is a continuous bijection from one compact space to another, it is bicontinuous. Thus Ξ is a homeomorphism between $\mathbf{X}$ and the metric space $\mathbf{Y}$. Hence $\mathbf{X}$ is metrizable.

There is a companion to Theorem A which concerns Baire sets instead of Baire functions. We state it below. Note that a collection $\{\mathbf{M}_\alpha\}$ of subset of a set X is said to separate the points of $\mathbf{X}$ providing that for every pair $x, y\in X$, $x\neq y$, there exists a set $\mathbf{M}_\alpha$ such that precisely one of the points, x, y belongs to $\mathbf{M}_\alpha$.

Corollary B. *Let* $\mathbf{X}$ *be a compact space. Then* $\mathbf{X}$ *is metrizable if and only if there exists a denumerable family* $\{\mathbf{M}_n\}$ *of Baire sets which separates the points of* $\mathbf{X}$.

Let χ_n be the characteristic function of $\mathbf{M}_n$. Then χ_n is a Baire function and the denumerable family $\{\chi_n\}$ separates the points of $\mathbf{X}$. Thus the preceding theorem applies. It should be noted that in case the family of Baire sets generates all the Baire sets, then, obviously, it separates points.

An important consequence of Theorem A will now be stated.

Corollary C. *If* $\mathbf{X}$ *is compact and is Baire isomorphic to a compact metric space* $\mathbf{Y}$, *then* $\mathbf{X}$ *is metrizable.*

Proof: Let $\Psi:\mathbf{X}\to\mathbf{Y}$ be the given Baire isomorphism. Let $\{g_1, g_2, \dots\}$ be a denumerable family of real-valued continuous functions defined on $\mathbf{Y}$ which separate the points of $\mathbf{Y}$. Define $\varphi_n = g_n \circ \Psi$, $n=1, 2, \dots$. Then φ_n is Baire measurable and hence a Baire function defined on $\mathbf{X}$ and the denumerable family $\{\varphi_1, \varphi_2, \dots\}$ separates the points of $\mathbf{X}$. Thus by Theorem A, $\mathbf{X}$ is metrizable.

A result including Corollary C is given by Jayne [1] for spaces $\mathbf{X}$ which enjoy certain properties in their beta compactification. The proof is non-elementary.

We introduce now a property that a topological space $\mathbf{X}$ may possess which plays a dominant role in what follows.

Property D. *There exits for each $x\in\mathbf{X}$ a sequence $(\mathbf{V}_n(x))$ of open neighborhoods of x such that if (x_n) is any sequence of points in $\mathbf{X}$, then $\bigcap_{n=1}^{\infty} \mathbf{V}_n(x_n)$ contains at most one point.*

Theorem E. *Let $\mathbf{X}$ be compact. Then $\mathbf{X}$ is metrizable if and only if $\mathbf{X}$ possesses property* D.

Proof. Suppose $\mathbf{X}$ is metric. For each x, let $\mathbf{V}_n(x)$ be the open sphere of radius $\frac{1}{n}$, center x. Then for any sequence (x_n), the intersection $\bigcap_{n=1}^{\infty} \mathbf{V}_n(x_n)$ contains at most one point. Hence $\mathbf{X}$ has property D.

We now assume that $\mathbf{X}$ has property D. We shall show that $\mathbf{X}$ is a Baire isomorphic to a compact metric space $\mathbf{Y}$ and is therefore metrizable by Corollary C.

For each $x\in\mathbf{X}$, let φ_x be a real-valued continuous function, $\varphi_x:\mathbf{X}\to[0, 1]$, such that $\varphi_x(x)=0$ and $\varphi_x(y)=1$ for $y\in\mathbf{X}-\mathbf{V}_1(x)$. Let

$$\mathbf{V}_1^0(x)=\left\{u:\ u\in\mathbf{X}\ \text{ and }\ \varphi_x(u)<\frac{1}{2}\right\};\quad \mathbf{F}_1(x)=\left\{u:\ u\in\mathbf{X}\ \text{ and }\ \varphi_x(u)\leqq\frac{1}{2}\right\}.$$

Thus $\mathbf{V}_1^0(x)\subset\mathbf{F}_1(x)\subset\mathbf{V}_1(x)$. The family $\{\mathbf{V}_1^0(x):x\in\mathbf{X}\}$ is an open cover for $\mathbf{X}$. Since $\mathbf{X}$ is compact, there exists a finite subcover in this family consisting, say, of $\{\mathbf{V}_1^0(x_1), \mathbf{V}_1^0(x_2), \dots, \mathbf{V}_1^0(x_{n_1})\}$. Thus $\{\mathbf{F}_1(x_1), \mathbf{F}_1(x_2), \dots, \mathbf{F}_1(x_{n_1})\}$ is also a cover of $\mathbf{X}$. We change the notation slightly and write $\{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_{n_1}\}$ for this cover. Thus

$$\mathbf{X} = \mathbf{F}_1\cup\mathbf{F}_2\cup\dots\cup\mathbf{F}_{n_1}.$$

Set $\mathbf{F}_n=\emptyset$ for $n>n_1$. Then $\mathbf{X}=\bigcup_{n=1}^{\infty}\mathbf{F}_n$. Now let $\mathbf{F}_1^*=\mathbf{F}_1$ and $\mathbf{F}_k^*=\mathbf{F}_k-(\mathbf{F}_1\cup\dots\cup\mathbf{F}_{k-1})$ for $k>1$. Thus $\mathbf{X}=\bigcup_{k=1}^{\infty}\mathbf{F}_k^*$ and the sets in the representation are disjoint.

Since the sets $\mathbf{F}_n$ are zerosets, $\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_k$ is also a zero set, $k=1, 2, \ldots$. Thus the set $\mathbf{F}_k^* = \mathbf{F}_k - (\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_{k-1})$ is a denumerable union of zerosets. Anticipating an inductive proceedure, we write k_1 instead of k. Thus for $k_1 > 1$ we have $\mathbf{F}_{k_1}^* = \bigcup_{r=1}^{\infty} \mathbf{H}_{k_1 r}$ where $\mathbf{H}_{k_1 r}$ is a zero set.

If $\mathbf{K}$ is any zeroset (hence compact), one may cover $\mathbf{K}$ by a finite collection of zerosets each of which lies inside the neighborhood $\mathbf{V}_2(x)$ for some point x. In particular, for the set $\mathbf{F}_1^*$, we have $\mathbf{F}_1^* = \bigcup_{k_2=1}^{\infty} \mathbf{F}_{1k_2}$ where the sets $\mathbf{F}_{1k_2}$ are zerosets of this type ($=\emptyset$ for large k_2).

We apply to the set $\mathbf{H}_{k_1 r}$ the argument just given for $\mathbf{K}$ and find a finite collection of zerosets, each contained in some $\mathbf{V}_2(x)$, whose union is $\mathbf{H}_{k_1 r}$. The totality of these sets for $r=1, 2, \ldots$ is then enumerated and labelled $\mathbf{F}_{k_1 k_2}$; $k_2=1, 2, \ldots$ and we obtain for $\mathbf{F}_{k_1}^*$ the expression $\mathbf{F}_{k_1}^* = \bigcup_{k_2=1}^{\infty} \mathbf{F}_{k_1 k_2}$ where each $\mathbf{F}_{k_1 k_2}$ is a zeroset lying inside some $\mathbf{V}_2(x)$.

Proceeding inductively we obtain for each integer $k_1 \geqq 1, k_2 \geqq 1, \ldots$ sets

$$\mathbf{F}_{k_1} \supset \mathbf{F}_{k_1}^* \supset \mathbf{F}_{k_1 k_2} \supset \mathbf{F}_{k_1 k_2}^* \supset \mathbf{F}_{k_1 k_2 k_3} \supset \mathbf{F}_{k_1 k_2 k_3}^* \supset \ldots$$

satisfying the relations:

$$\mathbf{X} = \bigcup_{k_1=1}^{\infty} \mathbf{F}_{k_1}; \quad \bigcup_{k_1=1}^{n} \mathbf{F}_{k_1} = \bigcup_{k_1=1}^{n} \mathbf{F}_{k_1}^*;$$

the sets $\mathbf{F}_{k_1}^*$ are disjoint;

$$\mathbf{F}_{k_1}^* = \bigcup_{k_2=1}^{\infty} \mathbf{F}_{k_1 k_2}; \quad \bigcup_{k_2=1}^{n} \mathbf{F}_{k_1 k_2} = \bigcup_{k_2=1}^{n} \mathbf{F}_{k_1 k_2}^*;$$

the sets $\mathbf{F}_{k_1 k_2}^*$ are disjoint, and so on. Furthermore, each set $\mathbf{F}_{k_1 k_2 \ldots k_n}$ lies inside the set $\mathbf{V}_n(x)$ for some x. Also, each set $\mathbf{F}_{k_1 \ldots k_n}$ is a zeroset and each set $\mathbf{F}_{k_1 \ldots k_n}^*$ is a denumerable union of zerosets. Clearly we have $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n} = \bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^*$.

Note that if x is given, then there exists a unique sequence of the type $(\mathbf{F}_{k_1}^*, \mathbf{F}_{k_1 k_2}^*, \ldots)$ such that x belongs to each set of the sequence. Note also that if $(k_1, k_2, \ldots)$ is any sequence of positive integers, then the intersection $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^*$ is non-empty if and only if each of the sets $\mathbf{F}_{k_1 \ldots k_n}^*$ is non-empty. This results from the fact that $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^* = \bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}$ and from compactness. Finally, since $\mathbf{F}_{k_1 \ldots k_n}^* \subset \mathbf{V}_n(x_n)$ for some point x_n, $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^* \subset \bigcap_{n=1}^{\infty} \mathbf{V}_n(x_n)$ and by Property D, this intersection contains at most one point. In recapitulation, the intersection

$\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1 \dots k_n}$ consists of precisely one point if and only if each set $\mathbf{F}^*_{k_1 \dots k_n}$ is non-empty.

As usual, denote the set of all sequences of positive integers $k=(k_1, k_2, \dots)$ by $\mathbf{N}^{\mathbf{N}}$. The topology and metric character of $\mathbf{N}^{\mathbf{N}}$ play a very important role in what follows. Each $x \in \mathbf{X}$ has associated to it a unique sequence $(k_1, k_2, \dots)$ such that $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1 \dots k_n} = \{x\}$. Thus this defines a map

$$\Phi : \mathbf{X} \to \mathbf{N}^{\mathbf{N}},$$

where $\Phi(x)=(k_1, k_2, \dots)$.

It will be shown that Φ is a bijection from $\mathbf{X}$ to $\Phi(\mathbf{X})$. Furthermore, it will be shown that $\Phi(\mathbf{X})$ is a closed subset of $\mathbf{N}^{\mathbf{N}}$ and that Φ is a Baire isomorphism.

If $\Phi(x)=\Phi(x')=(k_1, k_2, \dots)$, then x and x' both belong to the set $\mathbf{F}^*_{k_1 \dots k_n}$, $n=1, 2, \dots$, hence both belong to $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1 \dots k_n}$. The latter consists of one point by Property D. Thus Φ is a bijection onto $\Phi(\mathbf{X})$.

Let $k \in \mathbf{N}^{\mathbf{N}}$, $k \notin \Phi(\mathbf{X})$. Suppose $k=(k_1, k_2, \dots)$. It is easy to see that $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1 \dots k_n} = \emptyset$. In fact, if the intersection were not empty, it would consist of a single point x by Property D and one would have $\Phi(x)=k$. Now, by compactness and monotonicity if $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1 \dots k_n} = \bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \dots k_n} = \emptyset$, then there exists n_0 such that $\mathbf{F}^*_{k_1 \dots k_n} = \emptyset$, $n \geqq n_0$. This implies that the neighborhood of k consisting of the points $(k_1, \dots, k_{n_0}, l_{n_0}+1, \dots)$ where l_{n_0+j} is arbitrary, $j=1, 2, \dots$, does not intersect $\Phi(\mathbf{X})$. In other words, $\Phi(\mathbf{X})$ is closed in $\mathbf{N}^{\mathbf{N}}$.

Next we show that Φ^{-1} is a continuous map. Let $k=(k_1, k_2, \dots)$ be a point of $\Phi(\mathbf{X})$ and suppose that $\Phi^{-1}(k)=x$. Let $\mathbf{V}(x)$ be any neighborhood of x. Then since $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1 \dots k_n} = \{x\}$, there exists by the compactness of X an integer n_0 such that $\mathbf{F}^*_{k_1 \dots k_{n_0}} \subset \mathbf{F}_{k_1 \dots k_{n_0}} \subset \mathbf{V}(x)$. Let $\mathbf{W}(k)=\mathbf{W}(k, n_0)=\{l=(l_1, l_2, \dots): l \in \Phi(\mathbf{X}),\ l_1=k_1, \dots, l_{n_0}=k_{n_0};\ l_{n_0+r} \in \mathbf{N},\ r=1, 2, \dots\}$. Then $\mathbf{W}(k)$ is an open neighborhood of k. Also, $\Phi^{-1}(\mathbf{W}(k)) \subset \mathbf{V}(x)$, since $\Phi^{-1}(\mathbf{W}(k)) = \mathbf{F}^*_{k_1 \dots k_{n_0}}$. Thus Φ^{-1} is continuous. It follows that Φ maps open sets into open sets. Thus Φ maps cozerosets into open sets (of a metric space) and hence Φ maps Baire sets into Baire sets.

Finally, we show that Φ^{-1} transforms Baire sets into Baire sets. Note first that the family of all sets $\mathbf{W}(k, n_0)$ where k is arbitrary and $n_0=1, 2, \dots$, is a denumerable base for the topology of $\Phi(\mathbf{X})$. Now $\Phi^{-1}(\mathbf{W}(k, n_0)) = \mathbf{F}^*_{k_1 \dots k_{n_0}}$. This set is a denumerable union of zerosets in $\mathbf{X}$. Thus if $\mathbf{G}$ is any open set in the metric space $\Phi(\mathbf{X})$, $\Phi^{-1}(\mathbf{G})$ is a Baire set in $\mathbf{X}$.

The above paragraphs prove that Φ is a Baire isomorphism. Also, $\Phi(X)$ is a complete separable metric space. If the cardinality of $\Phi(X)$ is $\mathbf{c}$ (cardinality of the continuum), then $\Phi(X)$ is Baire isomorphic to [0, 1], a compact metric space [2, p. 358]. If the cardinality of $\Phi(X)$ is $\aleph_0$, it is obviously Baire isomorphic to some compact metric space. Thus, in either case, X is metrizable by Corollary C. (For simplicity of statement, we invoke above the very general classical theorem concerning the Baire isomorphism. It would have been simple in our case to avoid the theorem and to construct the needed isomorphism directly.)

A different proof of Theorem E could be obtained by using the Urysohn metrization theorem. However, the development presented above will be used on subsequent occasions.

Added by the authors on May 2, 1973: As indicated in our introduction, the purpose of the paper is to provide a procedure for "grading" compact spaces. In this grading, the metric spaces occupy the position of simplest structure because they have Property D. The refinement procedure necessary to handle the non-metric case will be developed on another occasion. We wish to point out that a characterization of metric compact spaces has been given by V. SNEIDER, Continuous images of Souslin and Borel sets. Metrization theorems, *Doklady Akad. Nauk S.S.S.R.* (N.S.) **50** (1945), 77—79; *M.R.* 14—782.

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