On products of Toeplitz operators

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Dedicated to Béla Sz.-Nagy on his sixtieth birthday

1. Introduction

Brown and Halmos show in [2] that if f and g are functions in L^∞ of the unit circle and T_f and T_g are the corresponding Toeplitz operators on H^2, then for the equality $T_f T_g = T_{fg}$ to hold it is necessary and sufficient that either $\bar{f}$ or g belong to H^∞. The sufficiency of the preceding condition has been recognized since Toeplitz operators were first studied; it forms the basis for the Wiener—Hopf factorization technique. The necessity of the condition tells us that the equality $T_f T_g = T_{fg}$ is rather special.

It is much less special, however, for the difference $T_f T_g - T_{fg}$ to be compact, and this circumstance has been useful in the spectral analysis of certain Toeplitz operators. Coburn [3] has shown that $T_f T_g - T_{fg}$ is compact if either f or g is continuous, and Gohberg and Krupnik [9] have shown that $T_f T_g - T_{fg}$ is compact if f and g are both piecewise continuous and have no common discontinuities. In the present paper, a sufficient condition for the compactness of $T_f T_g - T_{fg}$ is presented which contains both of the conditions just mentioned.

For f and g in L^∞ and λ a point on the unit circle, we define

$$\mathrm{dist}_\lambda (f, g) = \mathrm{ess.} \limsup_{\substack{z \to \lambda \\ |z|=1}} |f(z) - g(z)|.$$

If we extend f and g harmonically into the unit disk by means of Poisson's formula, we can also write

$$\mathrm{dist}_\lambda (f, g) = \limsup_{\substack{z \to \lambda \\ |z|<1}} |f(z) - g(z)|.$$

For f in L^∞ we define

$$\mathrm{dist}_\lambda (f, H^\infty) = \inf \{\mathrm{dist}_\lambda (f, h): \ h \in H^\infty\}.$$

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A simple normal families argument shows that there exists an h in H^∞ such that $\text{dist}_\lambda(f, h)=\text{dist}_\lambda(f, H^\infty)$.

Theorem. *If f and g are in L^∞, and if for each λ on the unit circle either* $\text{dist}_\lambda(\bar{f}, H^\infty)=0$ *or* $\text{dist}_\lambda(g, H^\infty)=0$, *then $T_f T_g - T_{fg}$ is compact.*

The proof, which rests ultimately on Coburn's condition, is given in Section 2. R. G. Douglas [4, Corollary 7. 52] has found an alternative proof which uses the theory of C^*-algebras.

The theorem says, roughly, that if the condition in the Brown—Halmos theorem holds locally, then the equality $T_f T_g = T_{fg}$ holds to within a compact perturbation. Because the condition in the Brown—Halmos theorem is a necessary as well as a sufficient one, it is natural to conjecture the converse of the above theorem. The converse, however, is false, as is shown by a counter example in Section 3. Section 4 contains a partial converse of the theorem.

2. Proof of the theorem

For f in L^∞, the function $\text{dist}_\lambda(f, H^\infty)$ is upper semicontinuous with respect to λ. This function therefore attains a maximum on the unit circle.

Lemma. *If f is in L^∞, then* $\text{dist}(f, H^\infty + C) = \max\{\text{dist}_\lambda(f, H^\infty) : |\lambda| = 1\}$.

Here, C denotes the space of continuous complex valued functions on the unit circle. The lemma is an immediate consequence of a result of Bishop and Glicksberg [8, p. 419] on sets of antisymmetry for function algebras (see [6] for a fuller explanation). To keep this paper as elementary as possible, a simple direct proof of the lemma is provided in Section 5.

Let f and g satisfy the hypotheses of the theorem. Choose $\varepsilon > 0$, and let A and B be the sets of points λ on the unit circle where $\text{dist}_\lambda(\bar{f}, H^\infty) \geqq \varepsilon$ and $\text{dist}_\lambda(g, H^\infty) \geqq \varepsilon$, respectively. The sets A and B are then closed and disjoint. Choose a nonnegative function u in C such that $u \leqq 1$, $u=0$ on A, and $u=1$ on B. Let $v = 1-u$. By Coburn's condition we have (letting $K_1, K_2, \ldots$ denote compact operators),

$$(1) \qquad T_f T_g = T_f T_u T_g + T_f T_v T_g = (T_{fu} + K_1) T_g + T_f (T_{vg} + K_2) = T_{fu} T_g + T_f T_{vg} + K_3.$$

Since $\text{dist}_\lambda(vg, H^\infty) < \varepsilon$ for all λ, it follows from the lemma that $\text{dist}(vg, H^\infty + C) < \varepsilon$. Hence, we can write $vg = h + w + \varphi$ where h is in H^∞, w is in C, and $\|\varphi\|_\infty < \varepsilon$. Because $T_f T_h = T_{fh}$ and $T_f T_w = T_{fw} + K_4$ (by Coburn's condition), we have

$$(2) \qquad T_f T_{vg} - T_{fvg} = T_f T_\varphi - T_{f\varphi} + K_4 = S_1 + K_4,$$

where $\|S_1\| \leqq \|f\|_\infty \|\varphi\|_\infty + \|f\varphi\|_\infty \leqq 2\varepsilon \|f\|_\infty$. Exactly the same reasoning gives

$$(3) \qquad T_{fu} T_g - T_{fug} = S_2 + K_5,$$

where $\|S_2\| \leqq 2\varepsilon \|g\|_\infty$. Combining (1)—(3), we obtain

$$T_f T_g = T_{fug} + T_{fvg} + S_1 + S_2 + K_6 = T_{fg} + S_1 + S_2 + K_6.$$

Hence, the distance of $T_f T_g - T_{fg}$ from the set of compact operators is at most $\|S_1\| + \|S_2\| \leqq 2\varepsilon(\|f\|_\infty + \|g\|_\infty)$. As ε can be chosen arbitrarily small, it follows that $T_f T_g - T_{fg}$ is compact. The theorem is proved.

3. Counter example to the converse

To obtain the desired counter example, we take $g = \bar{\psi}$ where ψ is the inner function $\exp\left(\frac{z+1}{z-1}\right)$. We take for f a real function in L^∞ with the following properties: (i) f is continuous except at $z=1$; (ii) $f/(1-z)$ is in L^2; (iii) f is not in $H^\infty + C$. We defer the construction of f until later. We note that $\mathrm{dist}_1(f, H^\infty) > 0$ by the lemma in Section 2. Also, it is easy to see that $\mathrm{dist}_1(g, H^\infty) = 1$. The condition of the theorem therefore fails at $\lambda = 1$. We show that, nevertheless, the operator $T_f T_g - T_{fg}$ is compact.

Because

$$(T_f T_g - T_{fg}) T_\psi = T_f T_{\bar{\psi}} T_\psi - T_{f\bar{\psi}} T_\psi = T_f T_{\bar{\psi}\psi} - T_{f\bar{\psi}\psi} = T_f - T_f = 0,$$

the operator $T_f T_g - T_{fg}$ annihilates the subspace ψH^2. Hence, it will be enough to show that the restriction of $T_f T_g - T_{fg}$ to $H^2 \ominus \psi H^2$ is compact. Also, the operator $T_g = T_{\bar{\psi}}$ annihilates $H^2 \ominus \psi H^2$, so we need only show that the restriction of T_{fg} to $H^2 \ominus \psi H^2$ is compact. We shall show that, actually, the transformation from $H^2 \ominus \psi H^2$ into L^2 of multiplication by fg is compact. Because multiplication by g sends $H^2 \ominus \psi H^2$ isometrically onto $\bar{\psi} H^2 \ominus H^2$, this amounts to showing that the transformation of $\bar{\psi} H^2 \ominus H^2$ into L^2 of multiplication by f is compact. Let S denote the latter transformation.

To prove the compactness of S, we introduce the isometry V of L^2 of the circle onto $L^2(-\infty, \infty)$ defined by

$$(Vh)(x) = \pi^{-1/2}(x+i)^{-1} h\left(\frac{x-i}{x+i}\right).$$

This isometry transforms the operator on L^2 of multiplication by f into the operator on $L^2(-\infty, \infty)$ of multiplication by $\varphi(x) = f\left(\frac{x-i}{x+i}\right)$. From (ii) it follows that φ is in $L^2(-\infty, \infty)$. Let W be the Fourier—Plancherel transformation of $L^2(-\infty, \infty)$ onto itself. Then W transforms the operator on $L^2(-\infty, \infty)$ of multiplication by φ into the operator of convolution with the square-integrable function $k = (2\pi)^{1/2} W\varphi$. Now it is easy to check that the combined transformation $U = WV$ sends $\bar{\psi} H^2 \ominus H^2$

onto $L^2(-1, 0)$ (regarded as the subspace of functions in $L^2(-\infty, \infty)$ vanishing off $(-1,0)$). Hence U takes S into the transformation of $L^2(-1,0)$ into $L^2(-\infty,\infty)$ of convolution with k, that is, into the integral operator with kernel $K(x, y)= = k(x-y)\,(-\infty<x<\infty,\ -1<y<0)$. The square-integrability of k implies the square-integrability of K, so the integral operator in question is a Hilbert—Schmidt operator. Therefore S is compact, as desired.

It remains to construct a function f with the required properties. For this we employ the notion of mean oscillation.

Let m denote normalized Lebesgue measure on the unit circle. For f in L^1 of the circle and I a subarc of the circle, define $av_I f=m(I)^{-1}\int_I f\,dm$ and

$$M(f, I) = m(I)^{-1}\int_I |f-av_I f|\,dm.$$

Further, define

$$M_r(f)=\sup\{M(f, I)\colon\ m(I)\leqq r\} \text{ for } 0<r\leqq 1, \text{ and } M_0(f)=\lim_{r\to 0} M_r(f).$$

The quantity $M_1(f)$ is called the mean oscillation of f, and in case $M_1(f)<\infty$ we say that f has bounded mean oscillation (or that f is in BMO). In case $M_0(f)=0$ we say that f has vanishing mean oscillation (or that f is in VMO). The class BMO has recently been studied by FEFFERMAN and STEIN [7], who have proved, among other results, that a function belongs to BMO if and only if it can be written as $u+\tilde{v}$ where u and v belong to L^∞ and $\tilde{v}$ is the conjugate function of v. We need here the less difficult half of this equivalence, the half that asserts $L^\infty+(L^\infty)^\sim\subset$ BMO. This inclusion is bounded in the sense that there is a positive constant c with the property $M_1(u+\tilde{v})\leqq c(\|u\|_\infty+\|v\|_\infty)$ for all u, v in L^∞ [7].

The class VMO also has a simple characterization; it consists of all functions $u+\tilde{v}$ with u and v in C. Here we require only the inclusion $C+\tilde{C}\subset$ VMO, which can be proved as follows. As the inclusion $C\subset$VMO is obvious, it will be enough to show that $\tilde{C}\subset$VMO. Let v belong to C, and choose $\varepsilon>0$. Then there is a trigonometric polynomial p such that $\|v-p\|_\infty<\varepsilon$. Since $\tilde{p}$ is continuous we have

$$M_0(\tilde{v}) = M_0(\tilde{v}-\tilde{p}) \leqq M_1(\tilde{v}-\tilde{p})\leqq c\|v-p\|_\infty<c\varepsilon.$$

Because ε is arbitrary this shows that $M_0(\tilde{v})=0$, as desired.

Now it is trivial that a real L^∞ function belongs to $H^\infty+C$ if and only if it belongs to $C+\tilde{C}$. Thus, to guarantee that the f we construct satisfies condition (iii), it will suffice to arrange that $M_0(f)>0$.

To construct f we introduce the subarcs $I_n=\{e^{i\theta}\colon 2^{-n}\leqq\theta\leqq 2^{-n}+5^{-n}\}$, $n=1, 2, \ldots$. We define f to be 0 off $\cup I_n$. On I_n we define f so that it is real, continuous, bounded in modulus by 1, vanishes at the endpoints of I_n, and satisfies

$$\int_{I_n} f\,dm=0,\quad \int_{I_n}|f|\,dm\geqq\frac{1}{2}m(I_n).$$

From the preceding equality and inequality we have $M_0(f)\geqq 1/2$, and thus (iii) holds. As (i) is obvious, it only remains to check (ii), which amounts to showing that the function $f(e^{i\theta})/\theta$ belongs to L^2. On I_n we have $|f/\theta|\leqq 2^n$, and so

$$\int_{I_n} |f/\theta|^2\, dm \leqq 2^{2n} m(I_n) = (2\pi)^{-1}(4/5)^n.$$

The square-integrability of f/θ is now obvious, and the construction is complete.

It appears that any necessary and sufficient condition, in terms of the structures of f and g, for the compactness of $T_f T_g - T_{fg}$ will have to take account of subtleties of the behavior of the Gelfand transforms of f and g on the fibers of the Gelfand space of L^∞.

4. A partial converse

The above theorem does have a converse of sorts.

Theorem. *If g is in L^∞ and if $T_h T_g - T_{hg}$ is compact for all h in H^∞, then g is in $H^\infty + C$.*

This result was first conjectured by R. G. Douglas, who has independently found the following proof.

Under the hypotheses of the theorem, if h is any function in H^∞ and ψ is any inner function, then the operator

$$T_{\bar\psi}(T_h T_g - T_{hg}) = T_{\bar\psi h} T_g - T_{\bar\psi hg}$$

is compact. As the functions $\bar\psi h$ are dense in L^∞ [5], we may conclude that $T_f T_g - T_{fg}$ is compact for all f in L^∞.

For f in L^∞, let Γ_f be the Hankel operator induced by f, that is, the operator from H^2 to $(H^2)^\perp$ of multiplication by f followed by projection onto $(H^2)^\perp$. A theorem of Hartman [10] (see also [1]) states that Γ_f is compact if and only if f belongs to $H^\infty + C$. Now a simple calculation shows that $T_f T_g - T_{fg} = -\Gamma_{\bar f}^* \Gamma_g$, and thus $\Gamma_f^* \Gamma_g$ is compact for all f in L^∞. Taking $f = g$ we conclude that $\Gamma_g^* \Gamma_g$ is compact, and hence that Γ_g is compact. Therefore g is in $H^\infty + C$ by Hartman's theorem, as desired.

5. Proof of the lemma

We present here a simple direct proof of the lemma of Section 2. The proof depends on the fact that $H^\infty + C$ is an algebra [4].

Let f belong to L^∞. It is obvious that $\mathrm{dist}_\lambda(f, H^\infty) \leqq \mathrm{dist}(f, H^\infty + C)$ for each λ, so it will suffice to show that $\mathrm{dist}(f, H^\infty + C) \leqq \max\{\mathrm{dist}_\lambda(f, H^\infty) : |\lambda| = 1\}$. Let M denote the preceding maximum. Choose $\varepsilon > 0$, and for each λ choose an h_λ

in H^∞ such that dist $(f, h_\lambda) < M+\varepsilon$. Because $\text{dist}_z(f, h_\lambda)$ is an upper semicontinuous function of z, there is for each λ an open subarc J_λ of the unit circle containing λ such that $\text{dist}_z(f, h_\lambda) < M+2\varepsilon$ for all z in J_λ. Choose a finite number of the subarcs J_λ that cover the unit circle. Denote these subarcs by $J_1, \dots, J_p$ and the corresponding functions h_λ by $h_1, \dots, h_p$. Choose a partition of unity $\{w_k\}_{k=1}^q$ of the unit circle subordinate to the cover $\{J_n\}_{n=1}^p$ and consisting of nonnegative functions in C. Thus $\sum_1^q w_k = 1$ everywhere, and for each k there is an $n(k)$ such that $J_{n(k)}$ contains the support of w_k. By the latter property, if $w_k(\lambda) \neq 0$ then $\text{dist}_\lambda(f, h_{n(k)}) < M+2\varepsilon$.

Now let $g = \sum_1^q w_k h_{n(k)}$. Then g is in $H^\infty + C$, and for any λ on the unit circle,

$$\text{dist}_\lambda(f, g) = \text{dist}_\lambda\Big(\sum_k w_k(\lambda) f, \sum_k w_k h_{n(k)}\Big) \leqq \sum_k \text{dist}_\lambda\big(w_k(\lambda) f, w_k h_{n(k)}\big) =$$

$$= \sum_k \text{dist}_\lambda\big(w_k(\lambda) f, w_k(\lambda) h_{n(k)}\big) = \sum_k w_k(\lambda)\, \text{dist}_\lambda(f, h_{n(k)}) < \sum_k w_k(\lambda)(M+2\varepsilon) = M+2\varepsilon.$$

It follows that $\|f-g\|_\infty < M+2\varepsilon$. We may conclude that dist $(f, H^\infty + C) \leqq M$, and the lemma is proved.

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