On a Fourier $L^1(E_n)$-multiplier criterion

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Dedicated to Professor B. Sz.-Nagy on the occasion of his 60th birthday

The main purpose of this note is to give a simple sufficient criterion for radial functions on the Euclidean n-space E_n to be the Fourier transform of an integrable function. The present criterion is a partial generalization of a well-known one-dimensional result due to Sz.-Nagy [9], namely

Theorem A. Let $h(v)$ be an even (continuous) function on $(-\infty, \infty)$ satisfying the following conditions i) $h(v) \to 0$ for $v \to \infty$, ii) $h'(v) \in L(0, \infty)$, iii) h' is locally of bounded variation except at the points $a_0 = 0 < a_1 < \cdots < a_s < \infty$ but in the neighbourhoods of a_i the integrals $\int\limits_{0+} v\,|dh'(v)|$,

$$\Big(\int^{a_i-} + \int_{a_i+}\Big) |v - a_i| \log (1/|v-a_i|)\,|dh'(v)| \quad (1 \leqq i \leqq s < \infty),$$

and $\int^{\infty} v\,|dh'(v)|$ converge. Then there exists an even integrable function H such that

$$h(v) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} H(x) e^{-ivx}\, dx.$$

The case $s=0$ is the one considered in [1], [2]; for further details see also [4; p. 251, p. 276].

Apart from a regularization at the critical points a_i, $1 \leqq i \leqq s$, introduced in the course of a partial integration, the proof mainly depends upon the absolute integrability of the Fejér-kernel on $(-\infty, \infty)$. For the n-dimensional analogue we will make heavy use of the absolute integrability of a suitable Riesz-kernel. This paper was written while the author held a DFG-fellowship; the author thanks Professor R. J. Nessel for a careful reading of the manuscript.

First let us give some notations. Let v, x, y denote elements of $E_n (x = (x_1, \dots, x_n))$, $x \cdot y = \sum_{k=1}^{n} x_k y_k$ the inner product, $|x| = x \cdot x^{1/2}$ the absolute value; let $m = (m_1, \dots, m_n)$

be an n-tuple of non-negative integers with $|m| = \sum_{k=1}^{n} m_k$, D^m the differential operator $(\partial/\partial x_1)^{m_1} \dots (\partial/\partial x_n)^{m_n}$, $[\alpha]$ the largest integer less than or equal to $\alpha \in E_1$. A function $f(x)$, defined on E_n, is called radial if $f(x)=f(|x|)$. Let the Fourier transformation on $\mathbf{L}^1(E_n)$, the set of all integrable functions on E_n, be defined by

$$[f]^\wedge(v) \equiv f^\wedge(v) = (2\pi)^{-n/2} \int_{E_n} f(x) e^{-iv\cdot x}\, dx$$

and let $[\mathbf{L}^1(E_n)]^\wedge$ be the set of all continuous functions which are equal to the Fourier transform of an $\mathbf{L}^1$-function.

For our multiplier theorem it is convenient to introduce the class $\mathbf{BV}_{j+1}$ consisting of those continuous functions h on $[0, \infty)$ such that $h, \dots, h^{(j-2)}$ are absolutely continuous on $(0, \infty)$, $h^{(j-1)}$ locally absolutely continuous, $\lim_{\tau\to\infty} h^{(i)}(\tau)=0$ for $0 \leqq i \leqq j-1$, and $h^{(j)}$ locally of bounded variation on $(0, \infty)$ with

$$\int_0^\infty \tau^j |dh^{(j)}(\tau)| < \infty. \tag{1}$$

It follows readily that

$$\mathbf{BV}_{j+1} \subset \mathbf{BV}_j. \tag{2}$$

Indeed, for $\varepsilon, R>0$ and $h \in \mathbf{BV}_{j+1}$ one has

$$\int_\varepsilon^R \tau^j\, dh^{(j)}(\tau) = \sum_{\nu=0}^{j} (-1)^\nu \frac{j!}{(j-\nu)!} \tau^{j-\nu} h^{(j-\nu)}(\tau)\Big|_\varepsilon^R$$

which, by hypothesis, remains bounded for $R\to\infty$. Observing that $\lim_{R\to\infty} h^{(j-\nu)}(R)=0$, $1 \leqq \nu \leqq j$, one necessarily has $h^{(j)}(R)\to 0$ for $R\to\infty$. Now Dirichlet's formula yields

$$\int_0^\infty \tau^{j-1} |h^{(j)}(\tau)|\, d\tau = \int_0^\infty \tau^{j-1} \Big| \int_\tau^\infty dh^{(j)}(\omega) \Big|\, d\tau \leqq$$

$$\leqq \int_0^\infty |dh^{(j)}(\omega)| \int_0^\omega \tau^{j-1}\, d\tau = j^{-1} \int_0^\infty \omega^j |dh^{(j)}(\omega)|.$$

The classes $\mathbf{BV}_{j+1}$ have already been considered in BUTZER—NESSEL—TREBELS [5] in order to obtain a simple estimate of

$$\sum_{k=0}^{\infty} \binom{k+j}{j} |\Delta^{j+1}\alpha_k| < \infty, \quad \Delta\alpha_k = \alpha_k - \alpha_{k+1}, \quad \Delta^{j+1} = \Delta\Delta^j, \tag{3}$$

the latter being a multiplier condition on a Banach space with a total sequence $\{P_k\}$ of orthogonal bounded linear projections under the hypothesis that $f \sim \sum P_k f$ is (C, j)-bounded. In this respect, the following theorem is the concrete continuous analogue of the abstract discrete multiplier theorem mentioned above. Indeed it is quite natural to replace the (C, j)-boundedness of the abstract Fourier expansion by the boundedness of the corresponding $(1, j)$-Riesz-means in case of Fourier integrals. Here the $(\varkappa, \lambda)$-Riesz-means are defined for $\varkappa, \lambda > 0$ on $\mathbf{S}$ (the set of infinitely differentiable, rapidly decreasing functions) by

$$R_{\varkappa,\lambda}(\varrho)f = \varrho^n r_{\varkappa,\lambda}(\varrho\,\cdot) * f, \quad [r_{\varkappa,\lambda}]^\wedge(v) = \begin{cases} (1-|v|^\varkappa)^\lambda, & |v| \leqq 1 \\ 0, & |v| \geqq 1 \end{cases} \tag{4}$$

where $*$ convolution, $r_{\varkappa,\lambda}$, and its Fourier transform are to be understood in the distributional sense. It is known (see e.g. [6]) that

$$r_{\varkappa,\lambda} \in \mathbf{L}^1(E_n) \quad \text{for} \quad \varkappa > 0, \quad \lambda > (n-1)/2; \tag{5}$$

thus, (4) is meaningful for all $f \in \mathbf{L}^1(E_n)$ for these $\varkappa$, λ-values and $[r_{\varkappa,\lambda}]^\wedge$ exists in the classical sense.

Theorem 1. *If* $h \in \mathbf{BV}_{j+1}$ *for* $j = [(n-1)/2]+1$, *then* $h(|v|) \in [\mathbf{L}^1(E_n)]^\wedge$.

Proof. Consider the function

$$H(x) = [(-1)^j/j!] \int_0^\infty \tau^{j+n} r_{1,j}(\tau x)\, dh^{(j)}(\tau)$$

which is integrable on account of the hypothesis and (5):

$$\int_{E_n} |H(x)|\, dx \leqq \int_0^\infty \tau^j |dh^{(j)}(\tau)| \int_{E_n} \tau^n |r_{1,j}(\tau x)|\, dx < \infty.$$

Passing to Fourier transforms, by Fubini's theorem and partial integration

$$H^\wedge(v) = ((-1)^j/j!) \int_0^\infty \tau^j \begin{Bmatrix} \left(1-\dfrac{|v|}{\tau}\right)^j, & |v| \leqq \tau \\ 0, & |v| \geqq \tau \end{Bmatrix} dh^{(j)}(\tau) =$$

$$= ((-1)^j/j!) \int_{|v|}^\infty (\tau - |v|)^j\, dh^{(j)}(\tau) =$$

$$= ((-1)^j/(j!)) \left\{ (\tau-|v|)^j h^{(j)}(\tau) \Big|_{|v|}^\infty - j \int_{|v|}^\infty (\tau-|v|)^{j-1} h^{(j)}(\tau)\, d\tau \right\}.$$

Now $h^{(j)}(\tau)$ is locally of bounded variation in $(0, \infty)$, and therefore the first term vanishes at $\tau = |v|$. Since $h^{(j)}(\tau) \to 0$ for $\tau \to \infty$, it follows that

$$|R^j h^{(j)}(R)| = \left| R^j \int_R^\infty dh^{(j)}(\tau) \right| \leqq \int_R^\infty \tau^j |dh^{(j)}(\tau)|$$

becomes infinitely small for $R\to\infty$. Hence

$$H^{\wedge}(v) = ((-1)^{j-1}/(j-1)!)\int_{|v|}^{\infty}(\tau-|v|)^{j-1}h^{(j)}(\tau)\,d\tau$$

and thus, proceeding iteratively,

$$H^{\wedge}(v) = -\int_{|v|}^{\infty}(\tau-|v|)h^{(2)}(\tau)\,d\tau = \int_{|v|}^{\infty}h'(\tau)\,d\tau = h(|v|).$$

Using (5) for arbitrary $\varkappa>0$, $\lambda=j=[(n-1)/2]+1$, it is clear by the above proof that $|v|$ may be replaced by $|v|^{\varkappa}$ provided $\int_0^{\infty}|v|^{\varkappa j}|dh^{(j)}(|v|^{\varkappa})|<\infty$ and $\lim_{\tau\to\infty}h^{(i)}(\tau)=0$ $(0\leqq i\leqq j-1)$ which, however, is equivalent to $h\in\mathbf{BV}_{j+1}$ on account of the homogeneity of the integral. Thus

Corollary. *If* $h\in\mathbf{BV}_{j+1}$ *for* $j=[(n-1)/2]+1$, *then* $h(|v|^{\varkappa})\in\mathbf{L}^1[(E_n)]^{\wedge}$ *for* $\varkappa>0$.

Obviously, Theorem 1 is a generalization of Sz.-Nagy's theorem in case $i=0$ to n-dimensions. In case there is a further singularity at the point $a\in(0,\infty)$ one could proceed analogously as in Sz.-Nagy's proof provided one can estimate $\{a^{j+n}r_{1,j}(ax)-\tau^{j+n}r_{1,j}(\tau x)\}$ in the $\mathbf{L}^1(E_n)$-norm conveniently, e.g. by $O(|a-\tau|^{\alpha})$ for some $\alpha>0$. However, we do not pursue this aspect further since there exists a convenient general multiplier theorem of Löfström [6, 7] dealing with such a finite number of singularities.

In case that singularities are admitted only at the origin and/or at infinity, Löfström's result was improved by Boman [3] to ($\mathbf{C}^N(A)$ the set of all N-times continuously differentiable functions on the open $A\subset E_n$):

Theorem B. a) If $f\in\mathbf{C}^N(E_n)$, where $N=[n/2]+1$, and there exist constants C and $\delta>0$ such that

$$|D^m f(x)|\leqq C|x|^{-\delta-|m|}\quad (x\in E_n,\quad 0\leqq|m|\leqq N),$$

then $f\in[\mathbf{L}^1(E_n)]^{\wedge}$.

b) Let $f\in\mathbf{C}^N(E_n\setminus\{0\})$, $N=[n/2]+1$, have compact support, and let there exist constants C and $\delta>0$ such that

$$|D^m f(x)|\leqq C|x|^{\delta-|m|}\quad (x\in E_n\setminus\{0\},\quad 0\leqq|m|\leqq N),$$

then $f\in[\mathbf{L}^1(E_n)]^{\wedge}$.

To illustrate the range of Theorems 1 and B, consider

$$f_1(x) = \{1+\log(1+|x|^2)\}^{-1}.$$

Obviously, f_1 is radial and belongs to $\mathbf{C}^{\infty}(E_n)$; but since f_1 decreases too weakly at infinity, Theorem B does not apply immediately, whereas a simple calculation shows that $f_1 \in \mathbf{BV}_{j+1}$. Thus $f_1 \in [\mathbf{L}^1(E_n)]^\wedge$ by Theorem 1. Analogously one has (cf. Corollary) $(1+\log\log(e+|x|^\varkappa))^{-\alpha} \in [\mathbf{L}^1(E_n)]^\wedge$ for $\varkappa, \alpha > 0$, etc.

To give an example with a singularity at the origin choose $f_2(x) = -\log^{-1}|x|\chi(|x|)$ with some $\chi \in \mathbf{C}^\infty(E_n)$ satisfying $\chi(x)=1$ for $0 \leqq |x| \leqq 1/e$ and $=0$ for $|x| \geqq 2/e$. Again, Theorem 1 yields $f_2 \in [\mathbf{L}^1(E_n)]^\wedge$, whereas Theorem B does not apply.

Naturally one could try "Bernstein's multiplier theorem" (see PEETRE [8]): $\dot{\mathbf{W}}^{n/2,1} \subset [\mathbf{L}^1(E_n)]^\wedge$, where $\dot{\mathbf{W}}^{n/2,1}$ may be equivalently characterized by

$$\int_0^\infty \tau^{-n/2} \sup_{|y| \leqq \tau} \|\Delta_y^n f(x)\|_2 \frac{d\tau}{\tau} < \infty$$

with $\Delta_y f(x) = f(x+y) - f(x)$. But to verify this condition in case of the above examples seems to be far harder than to check that $f \in \mathbf{BV}_{j+1}$ (other characterizations of $\dot{\mathbf{W}}^{n/2,1}$, known to the author, seem to be still more complicated).

The obvious disadvantage of Theorem 1 lies in the assumption that f has to be radial. Here another criterion, overlapping with Theorem 1 and but in some examples stronger than Theorem B, may help. Its proof rests upon the integrability of the Riesz-kernel $r_{\varkappa,1}$ on E_1 for $\varkappa > 0$, so that the product kernel $\prod_{k=1}^n r_{\varkappa_k,1}(x_k)$ is integrable on E_n. Thus

Theorem 2. *Let f be a continuous function on E_n, even in each coordinate, differentiable in the sense that for $0 \leqq m_i \leqq 2$ the derivatives $D^m f(x)$ exist as locally integrable functions, that $\lim_{x_k \to \infty} D^m f(x) = 0$ for $m_k = 0$ or 1 or 2, $1 \leqq k \leqq n$ and that*

$$\int_0^\infty \cdots \int_0^\infty |D^m f(x)| \prod_{m_k \neq 0} x_k^{m_k - 1} dx_k < \infty$$

uniformly in x_k when $m_k = 0$. Then $f(|x_1|^{\varkappa_1}, \ldots, |x_n|^{\varkappa_n}) \in [\mathbf{L}^1(E_n)]^\wedge$ provided $\varkappa_k > 0$, $1 \leqq k \leqq n$.

For the proof consider

$$F(y) = \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^n x_k^{1+(1/\varkappa_k)} r_{\varkappa_k}(x_k^{1/\varkappa_k} y_k) f^{(2n)}_{x_1 x_1 \ldots x_n x_n}(x)\, dx,$$

which is clearly integrable, and proceed as in the proof of Theorem 1.

Theorem 2 is another generalization of Sz.-Nagy's theorem [9]; in case $\varkappa_k = 1$, $1 \leqq k \leqq n$, his estimate of $\{a^2 r_{1,1}(a\eta) - \tau^2 r_{1,1}(\tau\eta)\}$, $a, \eta \in (0, \infty)$ in the $\mathbf{L}^1(-\infty, \infty)$-norm may be taken over to cover singularities on the hyperplanes $x_k = a > 0$, $1 \leqq k \leqq n$. Thus *a* theorem may be stated which is analogous to Theorem A. But instead of formulating it, let us give an example to which Theorem 2 applies but Theorem B does not since i) the function decreases at infinity too slowly, ii) Theorem B allows only a singularity at the origin and not on the hyperplanes $x_k=0$, $1 \leqq k \leqq n$. It is

$$\left(1+\log\left(1+|x_1|^{\varkappa_1}+\cdots+|x_n|^{\varkappa_n}\right)\right)^{-n} \in [\mathbf{L}^1(E_n)]^{\wedge}$$

provided $\varkappa_k>0$, $1 \leqq k \leqq n$, as can easily be shown by Theorem 2.

Let us conclude with the remark that Theorems 1 and 2 are based upon summability properties of the Fourier integral in direct analogy to the abstract series case as elaborated in [5].

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