Spectra of finite range Cesàro operators

By GERALD M. LEIBOWITZ in Storrs (Connecticut, U.S.A.)

In [BHS] BROWN, HALMOS, and SHIELDS studied the operators C_0, C_1, C_∞ defined respectively on the spaces l^2, $L^2(0, 1)$, $L^2(0, \infty)$, by

$$C_0 x(n) = \frac{1}{n+1} \sum_{k=0}^{n} x(k), \quad C_1 x(t) = \frac{1}{t} \int_0^t x(s)\, ds, \quad C_\infty x(t) = \frac{1}{t} \int_0^t x(s)\, ds.$$

In particular they determined, using Hilbert space techniques, that the adjoint of $I - C_1$ is a simple unilateral shift and the adjoint of $I - C_\infty$ is a simple bilateral shift, from which it follows that the spectrum of C_1 is the disk $\{\lambda : |1-\lambda| \leqq 1\}$ (with point spectrum the open disk $\{\lambda : |1-\lambda| < 1\}$) and the spectrum of C_∞ is the circle $\{\lambda : |1-\lambda| = 1\}$ (with point spectrum empty). We should point out that the fact that $I - C_\infty^*$ is unitary and has spectrum the unit circle can be obtained in another way, following ideas of GOLDBERG [G]. After mapping $L^2(0, \infty)$ isometrically onto the L^2 space of the multiplicative group G of positive real numbers (with respect to the Haar measure $\frac{dt}{t}$) via $x(t) \to t^{\frac{1}{2}} x(t)$, we see that C_∞^*, which is given by $C_\infty^* x(t) = \int_t^\infty \frac{x(s)}{s}\, ds$, is the operation of convolution by a certain function $\varphi \in L^1(G)$. Using the usual notation for Fourier transforms, one computes directly that $1 - \hat{\varphi}$ has modulus identically 1 and has each point of the unit circle in its essential range. It follows at once that $I - C_\infty^*$ is unitarily equivalent to an operator which is unitary and has the entire unit circle as its spectrum.

In [Bo], BOYD used an explicit integral formula for the resolvent to show that the corresponding operator T_∞ on $L^p(0, \infty)$ is a bounded operator mapping $L^p(0, \infty)$ into itself and having spectrum the circle $\left\{\lambda : \operatorname{Re} \frac{1}{\lambda} = \frac{p-1}{p}\right\}$ for $1 < p \leqq \infty$ (with $\frac{p-1}{p}$ defined to be 1 if $p = \infty$).

Here we determine the spectrum of the corresponding operator T_1 on the space $L^p(0, 1)$ $(1 < p \leqq \infty)$ and add a few remarks concerning T_∞.

Theorem. *Let* $1<p<\infty$ *and let* $(T_1x)(t) = t^{-1}\int_0^t x(s)ds$ for $x\in L^p(0, 1)$. *Then* T_1 *is a bounded linear operator on* $L^p(0, 1)$. *The spectrum of* T_1 *is the closed disk* $D_p = \left\{\lambda\colon \mathrm{Re}\frac{1}{\lambda} \geqq \frac{p-1}{p}\right\}$. *Each eigenvalue of* T_1 *has multiplicity* 1, *and the point spectrum of* T_1 *is the interior of* D_p.

Proof. By HARDY's inequality for integrals [HLP, p. 240], if $y\in L^p(0, \infty)$ then $T_\infty y\in L^p(0, \infty)$ and $\|T_\infty y\|_p < \frac{p}{p-1}\|y\|_p$ unless $y=0$ a.e. Hence T_∞ is a bounded operator on $L^p(0, \infty)$, and since the constant is best possible, $\|T_\infty\|_p = \frac{p}{p-1}$. From this it follows that T_1 is a bounded operator on $L^p(0, 1)$ with norm at most $\frac{p}{p-1}$. For if $x\in L^p(0, 1)$ and $\tilde{x}(t)=x(t)\,(0<t<1)$, $\tilde{x}(t)=0\,(t\geqq 1)$, then

$$\|T_1x\|_p=\left(\int_0^1 |T_\infty\tilde{x}(t)|^p\,dt\right)^{1/p} \leqq \|T_\infty\tilde{x}\|_p \leqq \frac{p}{p-1}\|\tilde{x}\|_p=\frac{p}{p-1}\|x\|_p.$$

We observe that if $x\in L^p(0, 1)$, then $x\in L^1(0, 1)$ and hence T_1x is a continuous function on $(0, 1)$. In particular, the range of T_1 is a proper subspace of $L^p(0, 1)$ so 0 belongs to the spectrum of T_1.

If $\lambda\neq 0$ and $T_1x=\lambda x$, it follows that x is continuous and hence by the fundamental theorem of calculus, that x is differentiable. Differentiating the relation $\lambda t x(t)= \int_0^t x(s)ds$, we have

$$\lambda t x'(t)+(\lambda-1)x(t) = 0.$$

This is an Euler differential equation of first order and thus its solutions have the form $x(t)=ct^\alpha$ where α is a complex scalar. We find $\lambda\alpha+(\lambda-1) = 0$ or $\alpha = \frac{1}{\lambda}-1$. (Thus, considered as a mapping from the space of integrable functions to the space of continuous functions on $(0, 1)$, T_1 has every nonzero number as a simple eigenvalue.) Since $t^\alpha\in L^p(0, 1)$ iff $\mathrm{Re}\,(\alpha p) > -1$, $t^{\frac{1}{\lambda}-1}\in L^p(0,1)$ iff $\mathrm{Re}\frac{1}{\lambda} > \frac{p-1}{p}$. So the point spectrum of T_1 is the interior of D_p and every eigenvalue of T_1 has geometric multiplicity 1. (Moreover, since $t^\alpha\notin L^p(0, \infty)$ for any α, the operator T_∞ has a void point spectrum.)

Next let the transformations P_ζ be defined by

$$(P_\zeta x)(t) = \int_0^1 s^{-\zeta} x(st)\, ds.$$

Then by Boyd's formula, P_ζ is a bounded operator on $L^p(0, 1)$ if $\operatorname{Re} \zeta < \frac{p-1}{p}$ and for such $\zeta, \zeta P_\zeta T_1 = \zeta T_1 P_\zeta = P_\zeta - T_1$. Hence if $\operatorname{Re} \frac{1}{\lambda} < \frac{p-1}{p}$ and $\zeta = \lambda^{-1}$, we see that $-\zeta^2 P_\zeta - \zeta I$ is a bounded operator inverse for $T_1 - \lambda I$; so λ belongs to the resolvent set for T_1. Thus $\sigma(T_1) \subset D_p$.

The spectrum of a bounded operator being compact, we must have $\sigma(T_1) = D_p$. We observe that the condition $\operatorname{Re} \frac{1}{\lambda} \geqq \frac{p-1}{p}$ is equivalent to the condition: $|\lambda|^2 \leqq \frac{p-1}{p} \operatorname{Re} \lambda$, so that D_p is the disk with center $\left(\frac{q}{2}, 0\right)$ and radius $\frac{q}{2}$ in R^2 (where $q = \frac{p-1}{p}$ is the conjugate index to p). Q.E.D.

The argument needs a slight modification when $p = \infty$. Since $t^\alpha \in L^\infty(0, 1)$ iff $\operatorname{Re} \lambda \geqq 0$, we find that $t^{\frac{1}{\lambda} - 1}$ is an eigenvector of T_1 on $L^\infty(0, 1)$ corresponding to the eigenvalue λ iff $\operatorname{Re} \frac{1}{\lambda} \geqq 1$. (Since $t^\alpha \in L^\infty(0, \infty)$ iff $\operatorname{Re} \alpha = 0$, the eigenvalues of T_∞ acting on $L^\infty(0, \infty)$ are the scalars λ with $\operatorname{Re} \frac{1}{\lambda} = 1$. Hence the operator T_∞ on $L^\infty(0, \infty)$ has spectrum which is entirely point spectrum.) Boyd's formula is still applicable, so $T_1 - \lambda I$ is invertible if $\operatorname{Re} \frac{1}{\lambda} < 1$. We summarize as follows.

Theorem. *Let T_1 be defined by the formula above. Then T_1 is a bounded linear operator on $L^\infty(0, 1)$. The spectrum of T_1 is the closed disk $D_\infty = \left\{\lambda : \operatorname{Re} \frac{1}{\lambda} \geqq 1\right\}$. The point spectrum of T_1 is $D_\infty \setminus \{0\}$ and each eigenvalue of T_1 has multiplicity* 1.

References

[BHS] A. Brown, P. Halmos, A. Shields, Cesàro operators, *Acta Sci. Math.*, **26** (1965), 125—137.
[Bo] D. Boyd, The spectrum of the Cesàro operator, *Acta Sci. Math.*, **29** (1968), 31—34.
[G] R. R. Goldberg, Watson transforms on groups, *Annals of Math.*, **71** (1960), 522—528.
[HLP] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press (1934).

THE UNIVERSITY OF CONNECTICUT

(Received November 4, 1971)