Spectra of convolution operators

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1. Introduction. A number of recent papers have dealt with the question of determining the spectrum of operators which are special cases of the following type of operator:

$$Tf(t) = \int_0^\infty k(s) f(ts)\, ds. \tag{1}$$

Here k is a given measurable function and the operator is considered as a mapping from $L^p(0, \infty)$ into itself. A sufficient condition for T to act as a bounded operator from $L^p(0, \infty)$ to itself is the well known result of HARDY, LITTLEWOOD and PÓLYA [6, p. 230] to the effect that

$$\|T\|_p \leqq \int_0^\infty |k(s)|\, s^{-1/p}\, ds = N_p(k) < \infty. \tag{2}$$

For example, BROWN, HALMOS and SHIELDS [2] by Hilbert space methods found the spectrum of the Cesàro operator

$$Pf(t) = \frac{1}{t} \int_0^t f(s)\, ds = \int_0^1 f(ts)\, ds. \tag{3}$$

In [1], this author gave an explicit formula for the resolvent of P as an operator on $L^p(0, \infty)$, $1 \leqq p \leqq \infty$, and from this deduced the spectrum of P. LEIBOWITZ [10] determined the spectrum of P as an operator on $L^p[0, 1]$. Recently, RHOADES [11] extended the considerations to operators corresponding to Gamma type summation methods. LEIBOWITZ [9] has determined the spectrum of operators of the type (1) where $k(s)$ vanishes for $s \geqq 1$, and for some $\varepsilon > 0$ satisfies

$$\int_0^1 k(s)\, s^{\varepsilon - (1/p)}\, ds < \infty. \tag{4}$$

Rhoades and Leibowitz also consider these operators as acting on $L^p(0, 1)$, and

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Leibowitz completely determines the spectrum in this case without the extra condition (4).

It is well known (see for example [12, p. 304, p. 311], [13, p. 36]) that operators of the type (1) are essentially convolution operators. This fact was used in [9]. Using this, it is clear that the spectrum of T as an operator on $L^p(0, \infty)$ is exactly the spectrum of the following operator $K*$ acting on $L^p(\mathbf{R})$:

$$K*u(x) = \int_{-\infty}^{\infty} K(x-y)\,u(y)\,dy, \tag{5}$$

where $K(x)=k(e^{-x})e^{-x/q}$, $(q = p/(p-1))$. The condition (2) translates into the condition $\|K*\|_p \leqq \|K\|_1$ which is a familiar inequality for convolutions [3, p. 528], [12, p. 97]. Note that the expression $\|K*\|_p$ denotes the operator norm of $K*$ acting on $L^p(\mathbf{R})$.

It is surely a familiar fact that the spectrum of $K*$ acting on $L^p(\mathbf{R})$ is the closure of the range of $\hat{K}$, the Fourier transform of K. Since we have been unable to locate a proof of this in the literature except for $p=1$ and 2, a proof is presented here as Theorem 1. From this it follows that the spectrum of T in $L^p(0, \infty)$ is the closure of the range of the Mellin transform

$$\hat{k}\left(-\frac{1}{p}+i\xi\right) = \int_{0}^{\infty} k(s)\,s^{-(1/p)+i\xi}\,ds. \tag{6}$$

For completeness, we also present some results concerning the point spectrum of convolution operators (Theorem 2) and point out that the Riesz—Thorin theorem produces an interesting inequality when applied to operators of type (1).

2. Convolution operators. In this section we will consider the operator $K*$ defined by (5) for $K \in L^1(\mathbf{R})$. We denote the Fourier transform of K by

$$\hat{K}(\xi) = \int_{-\infty}^{\infty} K(x)\,e^{i\xi x}\,dx. \tag{7}$$

We will always assume that $1 \leqq p \leqq \infty$. The spectrum of a bounded operator from a Banach space X into itself will be denoted by $\sigma(T; X)$.

The following deep result is due to WIENER and now usually established within the framework of the theory of Banach Algebras. See [4, p. 107] for a proof.

Lemma 1. *Let $K \in L^1(\mathbf{R})$ and suppose that λ is a complex number such that $\lambda \neq 0$, and $\lambda \neq \hat{K}(\xi)$ for any $\xi \in \mathbf{R}$. Then there is a function $A_\lambda \in L^1(\mathbf{R})$ such that*

$$\lambda A_\lambda - K * A_\lambda = K. \tag{8}$$

Corollary 1. *The spectrum of $K*$ as an operator on $L^p(\mathbf{R})$ is contained in the closure of the range of $\hat{K}$ on $\mathbf{R}$.*

Proof. If λ is not in the given set then by Lemma 1, there is an $A_\lambda \in L^1(\mathbf{R})$ satisfying (8). Since convolution is a commutative operation, one readily verifies that the operator $\lambda^{-1}(I+A_\lambda *)$, which is a bounded operator on $L^p(\mathbf{R})$, is the inverse of $(\lambda - K*)$, so λ is in the resolvent set of $K*$.

Lemma 2. *Let $1<p<\infty$. Let $K\in L^1(\mathbf{R})$ and suppose that*

$$\int_{-\infty}^{\infty} |xK(x)|\,dx = M < \infty. \tag{9}$$

Then, for each $\xi \in \mathbf{R}$, and $\delta>0$, there are functions $u_\delta \in L^p(\mathbf{R})$ of unit norm such that

$$\|\hat{K}(\xi)u_\delta - K*u_\delta\|_p = 0(\delta) \quad \text{as} \quad \delta \to 0.$$

Proof. For any $\delta>0$ and $\xi\in\mathbf{R}$, let

$$v_\delta(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}\,d\eta = 2e^{-i\xi x}(\sin \delta x)/x. \tag{10}$$

Then $v_\delta \in L^p(\mathbf{R})$ for $1<p<\infty$ and

$$\|v_\delta\|_p = \delta^{1-(1/p)}\|v_1\|_p. \tag{11}$$

Also, by interchange of order of integration, we have

$$K*v_\delta(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}\hat{K}(\eta)\,d\eta. \tag{12}$$

Thus

$$E_\delta(x) = \hat{K}(\xi)v_\delta(x) - K*v_\delta(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}(\hat{K}(\xi)-\hat{K}(\eta))\,d\eta. \tag{13}$$

The assumption (9) means that $\hat{K}'(\eta)$ exists and $|\hat{K}'(\eta)| \leqq M$ for all ξ, and with (13) this gives

$$|E_\delta(x)| \leqq M\delta^2. \tag{14}$$

We need a slightly better estimate than this for large x which we obtain from (13) by integration by parts, obtaining

$$|E_\delta(x)| \leqq 4M\delta x^{-1}. \tag{15}$$

Hence

$$\int_{-\infty}^{\infty} |E_\delta(x)|^p\,dx \leqq \int_{|x|<4/\delta} (M\delta^2)^p\,dx + \int_{|x|>4/\delta} (4M\delta x^{-1})^p\,dx = 0(\delta^{2p-1}). \tag{16}$$

Define $u_\delta = v_\delta/\|v_\delta\|$, and use (11) and (16) to complete the proof.

Theorem 1. *Let $K\in L^1(\mathbf{R})$. Let $1<p<\infty$. Then the spectrum of $K*$ as an operator on $L^p(\mathbf{R})$ is the closure of the range of $\hat{K}$ on $\mathbf{R}$.*

Proof. Denote the closure of the range of $\hat{K}$ by σ. By Corollary 1 the spectrum is contained in σ. It suffices then to show that if $\lambda=\hat{K}(\xi)$ for some $\xi\in\mathbf{R}$ then λ is in the spectrum of $K*$. If $p=\infty$, the function $e^{-i\xi x}$ is an eigenvector of $K*$ with eigenvalue $\hat{K}(\xi)$, proving that $\sigma(K*;L^\infty)=\sigma$. For $p=1$, the result follows by taking adjoints reducing to $p=\infty$. Finally if $1<p<\infty$, we use Lemma 2 as follows: let $K_n(x)=K(x)$ if $|x|\leqq n$ and zero otherwise. Then K_n satisfies (9) so there is a $u_n\in L^p$ with $|u_n|=1$ and

$$\|K_n*u_n-\hat{K}_n(\xi)\,u_n\|_p<1/n. \tag{17}$$

We also have

$$\|K_n*-K*\|_p\leqq\|K_n-K\|_1=\int\limits_{|x|\geqq n}|K(x)|\,dx=\varepsilon_n, \tag{18}$$

and

$$|\hat{K}_n(\xi)-\hat{K}(\xi)|\leqq\varepsilon_n, \tag{19}$$

where $\varepsilon_n\to0$ as $n\to\infty$.

Combining (17), (18), (19), we find that

$$\|K*u_n-\hat{K}(\xi)\,u_n\|_p\to0\quad\text{as}\quad n\to\infty \tag{20}$$

which shows that $\hat{K}(\xi)\in\sigma\big(K*;L^p(\mathbf{R})\big)$.

Corollary 2. Let k be a measurable function on $(0,\infty)$ satisfying

$$\int\limits_0^\infty|k(s)|\,s^{-1/p}\,ds<\infty. \tag{21}$$

Let T be defined as in (1) and $\hat{k}$ as in (6). Then the spectrum of T as an operator on $L^p(0,\infty)$ is the closure of the range of $\hat{k}\left(-\frac{1}{p}+i\xi\right)$ as ξ varies over $\mathbf{R}$.

Proof. Let $K(x)=k(e^{-x})e^{-x/q}$ where $q=p/(p-1)$. Then $K\in L^1(\mathbf{R})$. For any $f\in L^p(0,\infty)$, let $Qf(x)=f(e^x)e^{x/p}$. Then Q is an isometry of $L^p(0,\infty)$ onto $L^p(\mathbf{R})$. Furthermore $QTQ^{-1}u(x)=K*u(x)$ for all $u\in L^p(\mathbf{R})$. Thus $\sigma\big(T;L^p(0,\infty)\big)=$ $=\sigma\big(K*;L^p(\mathbf{R})\big)$ which is the closure of the range of $\hat{K}$ by Theorem 1. However, $\hat{K}(\xi)=\hat{k}\left(-\frac{1}{p}+i\xi\right)$.

Remarks 1. The proof of Theorem 1 for $p=1$ could be accomplished by noting that the norm of the operator $K*$ on $L^1(\mathbf{R})$ is the same as the L^1 norm of the function K, so the algebra of operators $K*$ is isometric with $L^1(\mathbf{R})$. The proof used for $1<p<\infty$ could also be modified to treat the case $p=1$.

2. Note that we made use of the Fourier transform only for $K\in L^1(\mathbf{R})$ and not for elements of $L^p(\mathbf{R})$. The usual proof of Theorem 1 for $p=2$ uses the fact

that the Fourier transform is a unitary operator on $L^2(\mathbf{R})$ so that $K*$ is unitarily equivalent to multiplication by $\hat{K}(\xi)$. Such a proof is not available for $p\neq 2$.

3. Our Corollary 2 contains the result of Leibowitz quoted in the introduction.

3. Point spectrum. Suppose that K satisfies the conditions of Theorem 1. The next theorem determines conditions under which a value λ will be in the point spectrum of $K*$ acting on $L^p(\mathbf{R})$. We denote the point spectrum by $\pi(K*;L^p)$. Our conditions are necessary and sufficient only in case $p=1, 2$ or ∞. In contrast to Theorem 1, we need the Fourier transform of elements of $L^p(\mathbf{R})$. We recall that if $1\leqq p\leqq 2$, and $u\in L^p$, then $\hat{u}\in L^q$, while if $2<p\leqq\infty$, $\hat{u}$ is a tempered distribution [7, p. 142 and p. 146]. The results of Theorem 2 can be translated into results for operators T of the form (1).

Theorem 2. *Let $K\in L^1(\mathbf{R})$ and for each complex number λ, let $E_\lambda=\{\xi:\hat{K}(\xi)=\lambda\}$. Then*

(a) *$\lambda\in\pi(K*;L^1)$ if and only if E_λ contains an interval,*

(b) *if $1<p<2$ and if E_λ is of measure zero, then $\lambda\notin\pi(K*;L^p)$, while if E_λ contains an interval then $\lambda\in\pi(K*;L^p)$,*

(c) *$\lambda\in\pi(K*;L^2)$ if and only if E_λ has positive measure,*

(d) *if $2<p<\infty$ and if E_λ is a finite set then $\lambda\notin\pi(K*;L^p)$, while if E_λ is of positive measure then $\lambda\in\pi(K*;L^p)$,*

(e) *$\lambda\in\pi(K*;L^\infty)$ if and only if E_λ is non-empty.*

Proof. Suppose that E_λ contains an interval $(a-\delta, a+\delta)$. Let $F(\xi)=\max(1-|\xi|,0)$ and $u(x)=\int_R F((\xi-a)/\delta)e^{-i\xi x}\,d\xi$. Then $u\in L^p(\mathbf{R})$ for all $p\geqq 1$, and since $F((\xi-a)/\delta)=0$ for $|\xi-a|>\delta$, we readily check that $K*u=\lambda u$. Similarly, if E_λ is of positive measure so contains a subset E of finite positive measure, then let $u(x)=\int_E e^{-i\xi x}\,d\xi$. Since χ_E is in L^q for $1\leqq q\leqq 2$, we have $u\in L^p$ for $2\leqq p\leqq\infty$, and as above, $K*u=\lambda u$. These remarks prove one direction of each of (a) to (d).

Conversely, suppose that u is in L^p and $K*u=\lambda u$. If $p=1$, this implies that

$$\hat{K}(\xi)\,\hat{u}(\xi)=\lambda\hat{u}(\xi) \tag{22}$$

for all ξ, and since $\hat{u}$ is continuous and vanishes except on E_λ by (22), it will vanish identically unless E_λ contains an interval. This proves (a), since $\hat{u}(\xi)=0$ for all ξ implies that $u=0$ a.e.

If $1<p\leqq 2$, equation (22) is valid a.e. so that $\hat{u}(\xi)$ vanishes for almost all $\xi\notin E_\lambda$, and hence vanishes a.e. if E_λ is of measure zero. By the uniqueness theorem $\hat{u}(\xi)=0$ a.e. implies that $u=0$ a.e. This completes the proof of (b) and (c).

If $2<p\leqq\infty$ and $K*u=\lambda u$ for $u\in L^p$, then (22) holds as a statement about tempered distributions. If φ is a testing function with support contained in an interval

I in the complement of E_λ, then there is a testing function ψ such that $(\hat{K}-\lambda)\psi = \varphi$. To see this, note that there is a $v \in L^1$ such that $\hat{v}(\xi) = (\hat{K}(\xi)-\lambda)^{-1}$ for $\xi \in I$ [5, p. 29]. Now let $\hat{\psi} = v * \hat{\varphi}$, and invert the Fourier transform to obtain ψ. Using

$$(\hat{K}-\lambda)\hat{u} = 0,$$

we have

$$0 = \langle(\hat{K}-\lambda)\hat{u}, \psi\rangle = \langle\hat{u}, (\hat{K}-\lambda)\psi\rangle = \langle\hat{u}, \varphi\rangle.$$

This shows that the support of $\hat{u}$ is contained in E_λ. If E_λ is finite then Theorem 4. 12 and Theorem 4. 11 of [7, p. 152] show that $\hat{u}$ is a finite linear combination of point measures. But then $u \notin L^p$ if $p<\infty$. This contradiction shows that $\lambda \notin \pi(K*; L^p)$ and completes the proof of (d).

The proof of (e) is left to the reader.

4. Norms. According to Corollary 1, the spectral radius of a T given by (1) as an operator on $L^p(0, \infty)$ is given by

$$r_p(T) = \max_{\infty<\xi<\infty} \left|\hat{k}\left(-\frac{1}{p}+i\xi\right)\right|.$$

This is also the norm of T in case $p=2$, since T is a normal operator. This can also be proved directly using the Fourier transform as in KOBER [8]. For $p=1$ or ∞, the norm of T is given by $N_p(k)$ of (2). If we associate with T the convolution operator $K*$ as in Corollary 2, then $N_p(k)=\|K\|_1=\|K*\|_1=\|K*\|_\infty$, and $r_p(T)=\max|\hat{K}(\xi)| = \|K*\|_2$. Thus the Riesz—Thorin convexity theorem shows that in general

$$\|T\|_p \leqq N_p(k)^\gamma r_p(T)^{1-\gamma} \quad \text{where} \quad \gamma = |2-p|/p. \tag{23}$$

5. An example. Let T defined by the following expression:

$$Tf(t) = \int_0^1 s^{-1+(2/p)} f(ts)\, ds - \int_1^\infty s^{-1} f(ts)\, ds.$$

Then $\hat{k}\left(-\frac{1}{p}+i\xi\right) = -2i\xi/(p^{-2}+\xi^2)$, and hence $r_p(T)=p$, and the spectrum of T on $L^p(0, \infty)$ is the set $\{i\eta: |\eta| \leqq p\}$. According to Theorem 2, there is no point spectrum if $p<\infty$, while if $p=\infty$, the whole spectrum consists of point spectrum. We do not know the value of $\|T\|_p$ but it is easy to compute $N_p(k)=2p$, and hence (23) gives the estimate

$$\|T\|_p \leqq 2^{|p-2|/p} p$$

and obviously $\|T\|_p \geqq r_p(T)=p$. It would be interesting to show that $\|T\|_p > p$ if $p \neq 2$.

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