Strictly cyclic shifts on l_p

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1. Introduction. Let X be a complex Banach space and let $\mathcal{A}$ be a closed abelian subalgebra of $\mathcal{B}(X)$, the algebra of all bounded linear transformations on X. $\mathcal{A}$ is said to be strictly cyclic if there is a vector x_0 in X such that $\mathcal{A}x_0=X$. General properties and examples of strictly cyclic algebras may be found in [1] and [4]. A large class of examples is given by the algebras generated by certain weighted shifts. In this paper we will be concerned with characterizing strictly cyclic weighted shifts on l_p. (An operator T is said to be strictly cyclic if the closed subalgebra it generates is strictly cyclic.)

For $1\leqq p<\infty$ let l_p be the Banach space of all absolutely p-summable sequences of complex numbers. Let $\{e_0, e_1, \dots\}$ be the standard basis for l_p. For each bounded sequence $\alpha=\{\alpha_1, \alpha_2, \dots\}$ of non-zero complex numbers the operator S_α in $\mathcal{B}(l_p)$ defined by $S_\alpha\left(\sum_{n=0}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} \alpha_n x_{n-1} e_n$ is called the weighted shift on l_p with weight sequence α. It is well known that $\|S_\alpha\|=\sup_n |\alpha_n|$. We set $\beta_0=1$ and $\beta_n=\alpha_1\alpha_2\dots\alpha_n$ for $n\geqq 1$. In [1] MARY EMBRY showed that S_α is strictly cyclic on l_1 if and only if $\sup_{n,m}\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right|<\infty$. Although this is not valid for $p>1$, we determine in this paper a dual result for shifts on l_p.

In § 2 we establish the basic notation and concepts used throughout the paper. Section 3 is concerned with strictly cyclic shifts on l_p for $1<p<\infty$. We give a general sufficient condition for strict cyclicity. Then shifts on l_p whose weights are monotone non-increasing in modulus are completely characterized. Several tests for strict cyclicity are given as corollaries to these results.

In § 4 weighted shifts on l_∞ are examined. The commutants of such shifts are characterized, with special emphasis on those shifts S_α such that $\inf_n |\alpha_n|>0$. To each weighted shift S_α on l_∞ there is associated in a natural way a closed abelian subalgebra $\mathcal{B}$ of the commutant of S_α. We obtain a necessary and sufficient condition for strict cyclicity for $\mathcal{B}$, analogous to our results for the case $p<\infty$. In conclusion we list a number of open questions relating to this material.

2. Preliminaries. The following facts about strictly cyclic abelian algebras will be used throughout this paper. Details can be found in [1]. Let x_0 be a strictly cyclic vector for the closed abelian subalgebra $\mathscr{A}$ of $\mathscr{B}(X)$. Then for each x in X there is a unique operator A_x in $\mathscr{A}$ such that $A_x x_0 = x$. The map $x \to A_x$ is a linear homeomorphism of X onto $\mathscr{A}$. Therefore there is a constant M such that for all x and y in X, $\|A_x y\| \leqq M\|x\|\,\|y\|$. We will concern ourselves with $X = l_p$.

For A in $\mathscr{B}(l_p)$ we let $\mathscr{A}(A)$ be the weakly closed subalgebra of $\mathscr{B}(l_p)$ generated by A and the identity operator I. That is, $\mathscr{A}(A)$ is the weak closure of the set of polynomials in A. We then let $\mathscr{A}'(A) = \{B \text{ in } \mathscr{B}(l_p): AB = BA\}$, called the commutant of A. It is well known that for every shift S_α on l_p, $\mathscr{A}'(S_\alpha)$ is a maximal abelian subalgebra of $\mathscr{B}(l_p)$ and e_0 is a cyclic vector for $\mathscr{A}(S_\alpha)$. Thus it follows from [3; Cor. 3. 3] that S_α is strictly cyclic if and only if e_0 is strictly cyclic for S_α. It is easy to see that any operator similar to a strictly cyclic operator is itself strictly cyclic, and that an argument completely analogous to [2; Th. 1] shows that S_α is similar, via an isometric isomorphism, to S_γ where $\gamma_n = |\alpha_n|$, $n = 1, 2, \dots$. Therefore, when convenient, we will assume our shifts to have positive weights.

Lemma 2. 1. *S_α is strictly cyclic on l_p if and only if*

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{n} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p < \infty \tag{1}$$

for all x and y in l_p.

Proof. Suppose S_α is strictly cyclic on l_p. Let y be in l_p and for each positive integer N let $A_N = \sum_{n=0}^{N} \frac{y_n}{\beta_n} S_\alpha^n$. Then A_N is in $\mathscr{A}(S_\alpha)$ and $\|(A_y - A_N)e_0\| = \left\| y - \sum_{n=0}^{N} y_n e_n \right\|$. Thus A_N converges in norm to A_y, and so $A_y = \sum_{n=0}^{\infty} \frac{y_n}{\beta_n} S_\alpha^n$, the series converging in the operator norm. Now, for each x and y in l_p,

$$A_y x = \sum_{n=0}^{\infty} \frac{y_n}{\beta_n} S_\alpha^n x = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_m y_n \frac{\beta_{n+m}}{\beta_n \beta_m} e_{n+m} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{\beta_n}{\beta_n \beta_{n-m}} x_m y_{n-m} \right) e_n .$$

Therefore, (1) holds.

Conversely, suppose (1) holds for each x and y in l_p. For each x in l_p let T_x be the linear transformation on l_p given by

$$T_x y = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right) e_n .$$

T_x is easily seen to be a closed linear transformation and so by the closed graph theorem T_x is bounded. Moreover, for each x and y in l_p, $T_x y = T_y x$. Thus $\|T_y x\| \leqq \leqq \|T_x\|\,\|y\|$ and by the uniform boundedness principle there exists a constant M such that $\|T_y x\| \leqq M\|x\|\,\|y\|$ for all x and y in l_p. For each non-negative integer N let

$y^{(N)} = \sum_{n=0}^{N} y_n e_n$. Then $\lim_{N\to\infty} \|T_y - T_{y^{(N)}}\| = 0$. But $T_{y^{(N)}} = \sum_{n=0}^{N} \frac{y_n}{\beta_n} S_\alpha^n$ and so T_y is a member of $\mathscr{A}(S_\alpha)$. Since $T_y e_0 = y$, S_α is strictly cyclic.

3. The case $1<p<\infty$. The following lemma gives the most general sufficient condition for strict cyclicity known to the authors at this time. This condition has been discovered independently by Mary Embry. Throughout this section we assume $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 3. 1. (Nikol'skiĭ [5]) *Let S_α be a weighted shift on l_p and suppose $M = \sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q < \infty$. Then S_α is strictly cyclic on l_p.*

Proof. Let x and y be in l_p. Then by Hölder's inequality,

$$\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{\beta_n}{\beta_m\beta_{n-m}}x_m y_{n-m}\right|^p \leqq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q\right)^{p/q}\left(\sum_{m=0}^{n}|x_m|^p|y_{n-m}|^p\right) \leqq$$

$$\leqq M^{p/q}\sum_{n=0}^{\infty}\sum_{m=0}^{n}|x_m|^p|y_{n-m}|^p = M^{p/q}\|x\|^p\|y\|^p.$$

By Lemma 2. 1, S_α is strictly cyclic.

We show now that under the assumption of monotonicity of the weights the converse to Lemma 3. 1 is valid.

Theorem 3. 2. *If $\{|\alpha_n|\}$ is monotonically non-increasing then S_α is strictly cyclic on l_p if and only if $\sup_n \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q < \infty$.*

Proof. Suppose S_α is strictly cyclic on l_p and $\{|\alpha_n|\}$ is monotonically non-increasing. By the remarks in section 2 we assume without loss of generality that each $\alpha_n>0$. By Lemma 2. 1 $\exists M>0$ such that $\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{\beta_n}{\beta_m\beta_{n-m}}x_m y_{n-m}\right|^p \leqq M\|x\|^p\|y\|^p$ for all x and y in l_p. Let x and y be in l_p with $x_n\geqq 0$ and $y_n\geqq 0$ for each n. For each positive integer N,

$$\sum_{n=N}^{2N}\left[\sum_{m=0}^{N}\frac{\beta_n}{\beta_m\beta_{n-m}}x_m y_{n-m}\right]^p \leqq M\|x\|^p\|y\|^p.$$

Since $\{\alpha_n\}$ is monotonically decreasing, $\frac{\beta_n}{\beta_{n-k}} \leqq \frac{\beta_m}{\beta_{m-k}}$ whenever $0\leqq k\leqq m\leqq n$. Therefore, replacing $\frac{\beta_n}{\beta_{n-m}}$ by $\frac{\beta_{2N}}{\beta_{2N-m}}$ in the above inequality we see that $\sum_{n=N}^{2N}\left[\sum_{m=0}^{N}\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}x_m y_{n-m}\right]^p \leqq M\|x\|^p\|y\|^p$. Let $y_k = \left(\frac{1}{2N+1}\right)^{1/p}$ for $0\leqq k\leqq 2N$ and $y_k=0$ otherwise. Then the preceding inequality reduces to

$$\frac{N+1}{2N+1}\left[\sum_{m=0}^{N}\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}x_m\right]^p \leqq M\|x\|^p.$$

Hence for every x in l_p, $\left|\sum_{m=0}^{N}\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}x_m\right| \leqq (2M)^{1/p}\|x\|$. It follows that

$$\sum_{m=0}^{N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq (2M)^{q/p} \equiv C.$$

Now,

$$\sum_{m=0}^{2N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q < \sum_{m=0}^{N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q + \sum_{m=N}^{2N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q = 2\sum_{m=0}^{N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq 2C.$$

On the other hand

$$\sum_{m=0}^{2N+1}\left(\frac{\beta_{2N+1}}{\beta_m\beta_{2N+1-m}}\right)^q = 1+\sum_{m=0}^{2N}\left(\frac{\beta_{2N+1}}{\beta_m\beta_{2N+1-m}}\right)^q =$$

$$= 1+\sum_{m=0}^{2N}\left(\frac{\alpha_{2N+1}}{\alpha_{2N+1-m}}\right)^q\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq 1+\sum_{m=0}^{2N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq 1+2C.$$

Thus we see that $\sup_n \sum_{m=0}^{n}\left(\frac{\beta_n}{\beta_m\beta_{n-m}}\right)^q \leqq 1+2C<\infty$, completing the proof.

Remark. The argument above is valid under the somewhat weaker assumption that $\{|\alpha_n|\}$ is ultimately monotone non-increasing.

Lemma 3. 1 and Theorem 3. 2 admit the following interesting corollaries. The first of these generalizes [4; Th. 4. 1].

Corollary 3. 3. *Suppose there exist u and v in l_q such that for all n and m, $\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right| \leqq |u_n|+|v_m|$. Then S_α is strictly cyclic on l_p.*

Proof. For $n\geqq m\geqq 0$, $\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q \leqq 2^q(|u_m|^q+|v_{n-m}|^q)$ and hence

$$\sup_n \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q \leqq 2^q(\|u\|_q^q+\|v\|_q^q)<\infty.$$

Corollary 3.4. *Suppose $\{|\alpha_n|\}$ is monotonically non-increasing and $\sum_{m=0}^{\infty}\left|\frac{\beta_{2m}}{\beta_m^2}\right|^q<\infty$. Then S_α is strictly cyclic on l_p.*

Proof. It is an easy consequence of the monotonicity assumption that for $i\geqq j\geqq 0$, $\left|\frac{\beta_{i+j}}{\beta_i\beta_j}\right| \leqq \left|\frac{\beta_{2j}}{\beta_j^2}\right|$ and so for any i and $j\geqq 0$, $\left|\frac{\beta_{i+j}}{\beta_i\beta_j}\right| \leqq \left|\frac{\beta_{2i}}{\beta_i^2}\right|+\left|\frac{\beta_{2j}}{\beta_j^2}\right|$. By Corollary 3. 3, S_α is strictly cyclic.

The next corollary shows that the collection of strictly cyclic shifts on l_p is fairly large.

Corollary 3. 5. *Suppose $\{|\alpha_n|\}$ is monotonically non-increasing and for some $r>0$, $\sum_{n=1}^{\infty}|\alpha_n|^r<\infty$. Then S_α is strictly cyclic on l_p for every $p\geqq 1$.*

Proof. Fix $p>1$ (the case $p=1$ follows from EMBRY's result mentioned above). We show that the hypothesis of Corollary 3. 3 is satisfied. Let M be a positive integer such that $Mq\geqq r$. Then if n and m are non-negative integers with $m\geqq M$ we have

$$\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right|=\frac{1}{|\beta_M|}|\alpha_{n+1}\cdots\alpha_{n+M}|\left|\frac{\alpha_{n+M+1}\cdots\alpha_{n+m}}{\alpha_{M+1}\cdots\alpha_m}\right|.$$

It follows from the assumption of monotonicity that

$$\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right|\leqq\frac{1}{|\beta_M|}|\alpha_{n+1}|^M.$$

Let $u_k=\max_{n,m<M}\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right|$ if $0\leqq k<M$ and $u_k=\frac{1}{|\beta_M|}|\alpha_{k+1}|^M$ otherwise. Then

$$\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right|\leqq u_n+u_m \text{ for all } n \text{ and } m\geqq 0.$$

Moreover since $Mq>r$, $\Sigma|\alpha_n|^{Mq}<\infty$ and consequently $\{u_n\}$ is in l_q. By Corollary 3. 3, S_α is strictly cyclic.

Only slight modifications of [3; Cor. 4. 8] show that if $\alpha_n=\frac{1}{\log(n+1)}$ for each $n\geqq 1$ then S_α is strictly cyclic on l_p for all $p>1$. However $\{\alpha_n\}$ decreases monotonically to 0 and is not r summable for any $r>0$.

We point out now a common theme in Embry's result concerning shifts on l_1 and our results for $p>1$. For $n\geqq 0$ let e_n' be the sequence $(0, 0, \ldots, 1, 0, 0, \ldots)$, the 1 in the n^{th} position (beginning the indexing at 0). For $p>1$ we consider $\{e_n'\}$ as the standard basis for l_q. For $p=1$ we may still write every element of l_∞ uniquely in the form $\sum_{n=0}^{\infty}a_n e_n'$. Now for $N\geqq 0$ let $f_N=\sum_{n=0}^{N}\frac{\beta_N}{\beta_n\beta_{N-n}}e_n'$, viewed as a continuous linear functional on l_p. Then for $p=1$, $\|f_N\|_\infty=\max_{0\leqq n\leqq N}\left|\frac{\beta_N}{\beta_n\beta_{N-n}}\right|$ and so Mary Embry's result can be rephrased in the following manner.

Theorem. (EMBRY) *S_α is strictly cyclic on l_1 if and only if $\{f_N\}$ is bounded.*

Our result above reduces to:

If $p>1$ and $\{f_N\}$ is bounded then S_α is strictly cyclic on l_p. The converse holds if $\{|\alpha_n|\}$ is monotonically non-increasing.

If $1<p<\infty$ and $\{|\alpha_n|\}$ is monotonically non-increasing then essentially the same proof as [6; Cor. 1] shows that the spectral radius r of S_α is $\lim_{N\to\infty} |\alpha_n|$. Now suppose $\{|\alpha_n|\}$ is monotonically non-increasing and S_α is strictly cyclic on l_p. Then $\{f_N\}$ is bounded in l_q. For fixed m and $N \geqq m$,

$$\langle f_N, e'_m\rangle = \frac{\beta_N}{\beta_m \beta_{N-m}} = \frac{1}{\beta_m}(\alpha_{N-m+1} \cdots \alpha_N),$$

hence for each m, $\lim_{N\to\infty} \langle f_N, e'_m\rangle = \dfrac{r^m}{\beta_m}$. It follows that f_N converges weakly to $\sum_{m=0}^{\infty} \dfrac{r^m}{\beta_m} e'_m$ in l_q.

We will see in the next section how some of these ideas may be extended to l_∞.

4. The case $p=\infty$. Since l_∞ is not separable there are no cyclic operators on l_∞. However for a bounded sequence $\{\alpha_n\}$ of complex numbers S_α still defines a bounded operator on l_∞ and we may ask when $\mathscr{A}'(S_\alpha)$ is strictly cyclic. First note that if T is a linear transformation from l_∞ to l_∞ then there is a sequence $\{t_0, t_1, \ldots\}$ of continuous linear functionals on l_∞ such that for every x in l_∞, $Tx = \langle t_0(x), t_1(x), \ldots\rangle$. Moreover, T is bounded if and only if $\sup_n \|t_n\| < \infty$. If this holds then $\|T\| = \sup_n \|t_n\|$. An easy computation shows that if T is a bounded linear operator on l_∞ with $\{t_n\}$ defined as above, then $TS_\alpha = S_\alpha T$ if and only if

(2) $$t_0 \circ S_\alpha = 0 \quad \text{and} \quad t_{n+1} \circ S_\alpha = \alpha_{n+1} t_n \qquad (n=0, 1, \ldots).$$

We now examine a special class of operators in $\mathscr{A}'(S_\alpha)$. Let

$$\mathscr{E} = \left\{x \text{ in } l_\infty \colon \sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| |x_m| < \infty\right\}.$$

For each x in $\mathscr{E}$ define the linear transformation A_x on l_∞ by

$$(A_x y)_n = \sum_{m=0}^{n} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \qquad (n=0, 1, 2, \cdots).$$

With $e_0 = (1, 0, 0, \ldots)$, etc., it is easily seen that $A_x e_0 = x$, $A_{e_1} = \dfrac{1}{\alpha_1} S_\alpha$, $A_{e_0} = 1$, A_x is bounded, and

$$\|A_x\| = \sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| |x_m|.$$

Let $\mathscr{B} = \{A_x \colon x \text{ in } \mathscr{E}\}$.

Lemma 4.1. *Let x and y be in $\mathscr{E}$. Then $z = A_x y$ is in $\mathscr{E}$ and $A_z = A_x A_y = A_y A_x$.*

Proof. We must show that $\sup_n \sum_{m=0}^{n} \left|\dfrac{\beta_n}{\beta_m \beta_{n-m}} z_m\right| < \infty$, where

$$z_k = \sum_{l=0}^{k} \frac{\beta_k}{\beta_l \beta_{k-l}} x_l y_{k-l}.$$

Fix an integer $n \geqq 0$. Then

$$\sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\sum_{k=0}^{m}\frac{\beta_m}{\beta_k\beta_{n-k}}x_k y_{m-k}\right| \leqq \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|\sum_{k=0}^{m}\left|\frac{\beta_m}{\beta_k\beta_{m-k}}\right||x_k||y_{m-k}| =$$

$$=\sum_{k=0}^{n}\left(\sum_{m=k}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|\left|\frac{\beta_m}{\beta_k\beta_{m-k}}\right||y_{m-k}|\right)|x_k| = \sum_{k=0}^{n}\left|\frac{\beta_n}{\beta_k\beta_{n-k}}\right|\left(\sum_{m=k}^{n}\left|\frac{\beta_{n-k}}{\beta_{n-m}\beta_{m-k}}\right||y_{m-k}|\right)|x_k| =$$

$$=\sum_{k=0}^{n}\left|\frac{\beta_n}{\beta_k\beta_{n-k}}\right|\left(\sum_{m=0}^{n-k}\left|\frac{\beta_{n-k}}{\beta_{n-k-m}\beta_m}\right||y_m|\right)|x_k| \leqq \|A_x\|\,\|A_y\|.$$

Techniques of rearrangement of series similar to those used above show that for each w in l_∞ and each non-negative integer n, $(A_z w)_n = (A_x A_y w)_n = A_y A_x w)_n$, i.e. $A_z = A_x A_y = A_y A_x$. Since $A_x + A_y = A_{x+y}$ we have proved part of the following result.

Theorem 4. 2. *$\mathscr{B}$ is a norm closed abelian subalgebra of $\mathscr{A}'(S_\alpha)$ with S_α and I in $\mathscr{B}$. Moreover, if $\inf_n |\alpha_n| > 0$, then $\mathscr{A}'(S_\alpha) = \mathscr{B}$.*

Proof. Let $\{x^{(N)}\}$ be a sequence in $\mathscr{E}$ and let A be a bounded operator on l_∞ such that $\lim_N \|A_{x^{(N)}} - A\| = 0$. Then $x^{(N)} = A_{x^{(N)}} e_0 \to x = Ae_0$. Choose $M > 0$ such that $\|A_{x^{(N)}}\| \leqq M$ for all N, i.e. $\sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right| |x_m^{(N)}| \leqq M$ for all N and $n \geqq 0$. Letting $N \to \infty$ we see that $\sup_n \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right| |x_m| \leqq M$ hence x is in $\mathscr{E}$ and $A_x = A$.

Now suppose $\delta = \inf_n |\alpha_n| > 0$. It is then immediate that the range of S_α^k is $\{x \text{ in } l_\infty : x_i = 0 \text{ for } 0 \leqq i \leqq k\}$. Let T be in $\mathscr{A}'(S_\alpha)$ with $\{t_n\}$ as in (2). Set $u = Te_0$. We show that u is in $\mathscr{E}$ and $A_u = T$. Let x be in l_∞ and write $x = x_0 e_0 + S_\alpha z$ for some z in l_∞. Then $t_0(x) = x_0 t_0(e_0) = x_0 u_0$. Now for $n \geqq 1$

$$t_n \circ S_\alpha^n = \alpha_n t_{n-1} \circ S_\alpha^{n-1} = \cdots = \beta_n t_0$$

so $t_n \circ S_\alpha^{n+1} = 0$ for all n. Then $t_n(x) = t_n\left(\sum_{m=0}^{n} x_m e_m\right) = \sum_{m=0}^{n} x_m t_n(e_m)$. Now for $m \geqq 1$,

$$t_n(e_m) = t_n\left(\frac{1}{\alpha_m} S_\alpha e_{m-1}\right) = \frac{1}{\alpha_m} t_n(S_\alpha e_{m-1}) = \frac{\alpha_n}{\alpha_m} t_{n-1}(e_{m-1}) = \cdots$$

$$\cdots = \frac{\alpha_n \cdots \alpha_{n-m+1}}{\alpha_m \cdots \alpha_1} t_{n-m}(e_0) = \frac{\beta_n}{\beta_m\beta_{n-m}} u_{n-m}.$$

Therefore $t_n(x) = \sum_{m=0}^{n} \frac{\beta_n}{\beta_m\beta_{n-m}} x_m u_{n-m}$. Thus u is in $\mathscr{E}$ and $A_u = T$, completing the proof.

Unlike the case $p < \infty$ the vector e_0 need not be cyclic for $\mathscr{A}'(S_\alpha)$. For example if $\alpha_n = 1$ for each n then by Theorem 4. 2, $\mathscr{A}'(S_\alpha) = \mathscr{B}$, and so $\mathscr{A}'(S_\alpha)e_0 = \mathscr{E}$. But in this case

$$\mathscr{E} = \left\{x \text{ in } l_\infty : \sum_{m=0}^{n} |x_m| < \infty\right\} = l_1$$

and hence is not dense in l_∞. However there are many shifts on l_∞ for which e_0 is in fact strictly cyclic for $\mathscr{B}$, and these can be classified precisely, in a manner analogous to Lemma 3. 1.

Proposition 4. 3. *The vector e_0 is strictly cyclic for $\mathscr{B}$ if and only if* $\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| < \infty$.

Proof. Suppose $M = \sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| < \infty$. Let x be in l_∞. Then

$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| |x_m| \leqq M \|x\|$$

and hence x is in $\mathscr{E}$, so that $\mathscr{E} = l_\infty$. Conversely, suppose $\mathscr{E} = l_\infty$. Then in particular for $x = (1, 1, 1, \cdots)$ in $\mathscr{E}$, we have $\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| < \infty$.

Some open questions:

1. Is the converse to Lemma 3. 1 valid?
2. If $\lim \alpha_n = 0$ need S_α be strictly cyclic on l_p, $1 < p < \infty$?
3. If S_α is a weighted shift on l_∞ is $\mathscr{A}'(S_\alpha)$ abelian?

Added in Proof. A negative answer to question 1 has recently been obtained by G. Fricke. Using Theorem 3. 2, R. Gellar and E. Azoff independently provided negative answers to question 2.

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