On p-pure subgroups of torsion-free cotorsion groups

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1. Introduction. In this paper we give an explicit form for the structure of torsion-free cotorsion groups (Theorem 1). We apply this to a special class of groups, the torsion-free abelian groups without elements of infinite p-height. A torsion-free abelian group G has an element a of infinite p-height if the equation $p^n x = a$ is solvable in G for any integer $n \geqq 1$ (p a prime). BOYER and MADER [5] have determined the structure of a torsion-free abelian group G without elements of infinite p-height in terms of p-pure and p-basic subgroups. With the aid of the torsion-free cotorsion groups we state the torsion-free part of their result more precisely (Theorem 2). Then we investigate the p-pure subgroups of groups G without elements of infinite p-height which have the additional property that G is complete with respect to the p-adic topology, the so-called p-closed groups. The similarity with the closed p-groups defined by FUCHS for the torsion case is obvious ([6], p. 114) and one can easily prove the analogues of theorems of p-groups for the torsion-free case (Lemma 3 and 4). Our main object is, however, to derive results on the extensions of homomorphisms for p-pure subgroups of torsion-free cotorsion groups. Our theorems 3 and 4 are generalizations of corresponding results of ARMSTRONG [1] for p-pure subgroups of the group of p-adic integers. Let S be a p-pure subgroup of a p-closed group G and let B be a p-basic subgroup of S. Then $\mathrm{Hom}\,(S/B, G/S)=0$ (or equivalently $\mathrm{Ext}\,(S/B, S)=0$) is a sufficient condition that every $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension to an endomorphism of S. Therefore we investigate the groups S with $\mathrm{Hom}\,(S/B, G/S)=0$. In Theorems 5 and 6 we give some equivalent statements for the condition $\mathrm{Hom}\,(S/B, G/S)=0$. It turns out that, if the rank of $S \leqq \aleph_0$, S is completely decomposable into a direct sum of copies of some torsion-free quotient-divisible group of rank 1. Finally we investigate the groups S as above but without the restriction $\mathrm{Hom}\,(S/B, G/S)=0$. In theorem 7 the structure of these groups is reduced to the case where S is a subgroup of $Z(p)$ containing 1 and with the property that S, as a ring, is a subring of $Z(p)$.

Our notation and terminology is, for the main part, in accordance with that of FUCHS [6]; for unexplained notions we refer to his book [6].

2. The structure of torsion-free cotorsion groups. Torsion-free cotorsion groups were defined by HARRISON [9]. For convenience let us summarize some results of [9]. The word group will always mean abelian group. The additive group of rationals is denoted by Q, the additive group of integers by Z. A group G is reduced if it has no non-trivial divisible subgroup. A reduced group G is called *cotorsion* if G a subgroup of a group M with M/G torsion-free imply that G is a direct summand of M, i.e. $\mathrm{Ext}(H, G)=0$ for all torsion-free groups H.

(i) There is a one-to-one correspondence between all divisible torsion groups and all torsion-free cotorsion groups. If D is a divisible torsion group, the correspondence is $D \to \mathrm{Hom}(Q/Z, D)$. If G is torsion-free cotorsion, the inverse of this correspondence is $G \to (Q/Z) \otimes G$.

A result of FUCHS [7, p. 123] states:

(ii) A torsion-free group is a cotorsion group if and only if it is a reduced algebraically compact group.

In [9, Prop. 2. 1., p. 371] it is proved:

(iii) A group is torsion-free cotorsion if and only if it is isomorphic to a direct summand of a complete (unrestricted) direct sum of p-adic integers.

Definition. The *height* of the p-adic integer π is the integer $k(\geqq 0)$ such that $\pi \in p^k Z(p)$, but $\pi \notin p^{k+1} Z(p)$, where $Z(p)$ is the group of p-adic integers. In order to find the structure of torsion-free cotorsion groups it is enough to determine the groups $\mathrm{Hom}(Q/Z, D)$ for arbitrary divisible torsion groups D (by (i)). The following theorem holds:

Theorem 1. *Let D be a divisible torsion group and suppose $D=\sum_{p_i} \sum_{\alpha_{p_i}} C(p_i^\infty)$, where $C(p_i^\infty)$ is the quasi-cyclic group of type p_i (p_i a prime). Then*

$$\mathrm{Hom}(Q/Z, D) \cong \sum_{p_j}^{*} \sum_{\alpha_{p_j}}' Z(p_j) \tag{1}$$

where the first (complete) sum Σ^* is taken over all prime numbers p_j and, for each prime number p_j, the number of components π_λ with height$=k$ in $\langle \dots, \pi_\lambda, \dots \rangle \in \sum_{\alpha_{p_j}} Z(p_j)$ is finite ($k=0, 1, 2, \dots$).

A proof of Theorem 1 is given in [11]. All torsion-free cotorsion groups have the structure (1) of an interdirect sum of groups of p-adic integers for different primes p and by the result of FUCHS [7, p. 123, (j)] the torsion-free reduced algebraically compact groups have this form. The following remarks are due to Prof. L. FUCHS.

We have $\Sigma Z(p) \subset \Sigma' Z(p) \subset \Sigma^* Z(p)$, where Σ'/Σ is the maximal divisible subgroup of Σ^*/Σ, the latter group being again algebraically compact. Actually, Σ' is the completion of Σ is the n-adic topology (cf. [9], p. 379), so Σ is dense is Σ' which means Σ'/Σ is divisible ([9], p. 380). Thus Σ^* will be the direct sum of Σ' and a reduced algebraically compact group $\cong \Sigma^*/\Sigma'$. Now we are going to use the concept of p-basic subgroup (p a prime), introduced in [7] by Fuchs. Let G be an arbitrary torsion-free abelian group. Then B_0 is called a *p-basic subgroup* of G, if the following conditions are satisfied:

(i) B_0 is a direct sum of infinite cyclic groups.

(ii) B_0 is a *p-pure subgroup* of G, i.e. $p^r B_0 = B_0 \cap p^r G$ for $r=0, 1, 2, \dots$.

(iii) The factor group G/B_0 is *p-divisible:* i.e. $p^n \bar{x} = \bar{a}$ is solvable in G/B_0 for any $\bar{a} \varepsilon G/B_0$ and any integer $n \geqq 0$.

In [7] it is shown that every torsion-free group G contains p-basic subgroups for every prime p. Moreover, the p-basic subgroups of G (for the same prime) are all isomorphic.

For each $\lambda \in \Lambda$ (the index set Λ is arbitrary) let $Z(p)_\lambda$ be the group of p-adic integers and Z_λ the infinite cyclic group of finite p-adic integers. Let $P = \sum_{\lambda \in \Lambda}^{*} Z(p)_\lambda$ be the complete direct sum and $R = \sum_{\lambda \in \Lambda} Z(p)_\lambda$ the discrete direct sum of the groups $Z(p)_\lambda$. If we introduce the n-adic (p-adic) topology for abelian groups (see [9]), then P is complete in the n-adic topology and the n-adic topology coincides with the p-adic topology. R is a pure subgroup of P, hence it possesses a completion in P for the coinciding n-adic and p-adic topologies. Let $C = (\sum_{\lambda} Z(p)_\lambda)^*$ be the completion of R in P, then, by the remarks of Fuchs, $C = \sum_{\lambda}' Z(p)_\lambda$ and C is a torsion-free cotorsion group. Moreover C is a direct summand of P.

Let G be an arbitrary torsion-free abelian group. One can define a homomorphism $\sigma: G \to P$ (into) such that the subgroup of elements of infinite p-height in G is the kernel of σ [5]. Assume now that G has no elements of infinite p-height. Then P contains an isomorphic copy $\sigma(G)$ of G. It is known that $\sigma(G)$ is a p-pure subgroup of P and hence $\sigma(G)$ possesses a p-adic completion in P. Let B be a p-basic subgroup of G, then $\sigma(B)$ is a p-basic subgroup of $\sigma(G)$. And $\sigma(B) \leqq \sigma(G)$ implies

(2) p-adic completion of $\sigma(B) \leqq p$-adic completion of $\sigma(G)$.

$\sigma(B)$ is dense in $\sigma(G)$ in the p-adic topology, hence $\sigma(G) \leqq p$-adic completion of $\sigma(B)$, which implies

(3) p-adic completion of $\sigma(G) \leqq p$-adic completion of $\sigma(B)$.

(2) and (3) imply that $\sigma(B)$ and $\sigma(G)$ have identical completions in the p-adic topology. We have proved:

Lemma 1. *Let G be a torsion-free group without elements of infinite p-height. If B is a p-basic subgroup of G, then B and G have identical completions is the p-adic topology.*

If B is a p-basic subgroup of G, then its isomorphic copy $\sigma(B)$ (in $\sigma(G)$) has the form $\sum_{\lambda \in \Lambda} Z_\lambda$ ([5]). As a direct consequence of Lemma 1 we get:

Lemma 2. *$R = \sum_\lambda Z(p)_\lambda$ and $\sigma(B) = \sum_\lambda Z_\lambda$ have the same p-adic completion in P.*

As we have seen the p-adic completion of R has the form $C = \sum'_\lambda Z(p)_\lambda$, so $\sigma(B)$ and $\sigma(G)$ have the same p-adic completion C by the lemma's 1 and 2. Also $\sigma(G)$ is p-pure in P implies $\sigma(G)$ is a p-pure subgroup of C. The torsion-free part of Corollary 2. 7 in [5] may be slightly sharpened in the following form:

Theorem 2. *Every torsion-free abelian group G without elements of infinite p-height may be considered to be a p-pure subgroup of some torsion-free cotorsion group $C = \sum'_\lambda Z(p)_\lambda$ and containing $B = \sum_\lambda Z_\lambda$ as a p-basic subgroup. C is the p-adic completion of G and B.*

According to a definition in [6], § 34, p. 114 for p-groups we define a torsion-free group G without elements of infinite p-height to be a *p-closed* group if every Cauchy sequence in G has a limit in G, i.e. if G is complete with respect to the p-adic topology. It is easy now to give results for p-closed groups which are analogous to the corresponding properties of p-groups. Here are 2 examples:

Lemma 3. *The torsion-free group G is p-closed if and only if G is the p-adic completion of a p-basic subgroup B of G (cf. Theorem* 34.1 *in* [6]*).*

Lemma 4. *Two p-closed groups are isomorphic if and only if their p-basic subgroups are isomorphic* (cf. Corollary 34. 2, [6], p. 115).

3. Extending homomorphisms. Now we apply the structure theorem 2 to the investigation of torsion-free abelian groups without elements of infinite p-height. We are able to generalize results of Armstrong [1] who obtained extension theorems for homomorphisms of p-pure subgroups of the group of p-adic integers. Let S be a p-pure subgroup of $G = \sum'_{\lambda \in \Lambda} Z(p)_\lambda$ and let S contain $B = \sum_\lambda Z_\lambda$ as a p-basic subgroup. Both G and B are fixed.

Now Hom $(G/S, G) = 0$, since the homomorphic image of a p-divisible group is again p-divisible. But G does not contain p-divisible subgroups $\neq 0$. Then

$$0 = \text{Hom}\,(G/S, G) \to \text{Hom}\,(G, G) \xrightarrow{j} \text{Hom}\,(S, G) \to \text{Ext}\,(G/S, G) \to \text{Ext}\,(G, G) = 0$$

is exact, where Ext $(G, G) = 0$ since G is a cotorsion group. The action of j is to restrict $\pi \in$ Hom (G, G) to S. Consequently, *every $\alpha \in$* Hom (S, G) *has an extension to an endomorphism of G if and only if* Ext $(G/S, G) = 0$.

In case there exists an extension $\bar{\alpha} \in$ Hom (G, G) of $\alpha \in$ Hom (S, G), then $\bar{\alpha}$ is uniquely determined, i.e. if $\bar{\alpha}$, $\bar{\beta}$ are endomorphisms of G which agree on S, then

$\bar{\alpha}=\bar{\beta}$. For S is contained in the kernel of the difference $\bar{\gamma}=\bar{\alpha}-\bar{\beta}$. Thus $\bar{\gamma}(G)$ is a homomorphic image of the p-divisible group G/S, and, for this reason, is p-divisible. Since G is p-reduced and since $\bar{\gamma}(G)\leqq G$, it follows that $\bar{\gamma}(G)=(\bar{\alpha}-\bar{\beta})(G)=0$. Thus $\bar{\alpha}=\bar{\beta}$.

Lemma 5. *Let L be a subgroup of a torsion-free group H and G an arbitrary torsion-free cotorsion group. Let L_* be the smallest pure subgroup of H containing L and p a rational prime. Then the following are equivalent:*

(1) *L is a p-pure subgroup of H.*

(2) *The p-primary component of the torsion-group L_*/L is* 0.

(3) Ext $(H/L, G)=0$.

(4) Ext $(L_*/L, G)=0$ (cf. Lemma [1], p. 317).

Proof. (1)$\leftrightarrow$(2) ([1], p. 317). Since L_* is pure in H and H is torsion-free, H/L_* is torsion-free. Hence Ext $(H/L_*, G)=0$, since G is cotorsion. Now $0=\text{Ext}\,(H/L_*, G)\to\to\text{Ext}\,(H/L, G)\to\text{Ext}\,(L_*/L, G)\to 0$ is exact, hence Ext $(H/L, G)=0\leftrightarrow$ Ext $(L_*/L, G)=0$ or (3)$\leftrightarrow$(4). Finally L_*/L is a torsion-group and G is torsion-free, so Ext $(L_*/L, G)\cong\cong$ Hom $(L_*/L, D/G)$, where D is the divisible hull of G. The maximal torsion subgroup T of D/G is a p-group and so Hom $(L_*/L, D/G)=$ Hom $(L_*/L, T)=$ Hom (p-primary component of L_*/L, T), which is zero if and only if the p-primary component of L_*/L is 0, since T is divisible. Hence Ext $(L_*/L, G)=0\leftrightarrow p$-primary component of L_*/L is 0 or (4)$\leftrightarrow$(2). This completes the proof.

Assume again that G is a torsion-free cotorsion group without elements of infinite p-height and let S be a subgroup of G. Then lemma 5 implies, taking $S=L$ and $H=G$, that S is a p-pure subgroup of G if and only if Ext $(G/S, G)=0$. Using our result above about the extension of homomorphisms we get a slight extension of a theorem of Armstrong:

Theorem 3. *Let G be a torsion-free cotorsion group without elements of infinite p-height (p-closed group). Let S be a subgroup of G, then the following are equivalent:*

(i) *S is a p-pure subgroup of G.*

(ii) Ext $(G/S, G)=0$.

(iii) *Every homomorphism of S into G may be extended to an endomorphism of G.* (cf. Theorem, [1], p. 318).

Every torsion-free abelian group S without elements of infinite p-height may be considered to be a p-pure subgroup of a p-closed group by theorem 2, hence such a group satisfies conditions (ii) and (iii).

Now the structure of Hom (G, G) for a p-closed group G can easily be derived. Let $G = \sum_m' Z(p)$. We know that $B = \sum_m Z$ is a p-basic subgroup of G. Hence B is a p-pure subgroup of G, but then Hom $(B, G) \cong$ Hom (G, G) by theorem 3. And Hom $(B, G) =$ Hom $(\sum_m Z, G) \cong \sum_m^*$ Hom $(Z, G) \cong \sum_m^* G$. Hence Hom $(G, G) \cong \sum_m^* G$ Now we are interested is the endomorphism groups of p-pure subgroups of G. First we prove: Let S and T be p-pure subgroups of $G = \sum_{\lambda \in \Lambda}' Z(p)_\lambda$. Then each element of Hom (S, T) may be extended uniquely to an endomorphism of G. Indeed, we know that Hom $(S, G) \cong$ Hom (G, G). Hence Hom (S, T) is a subgroup of Hom (G, G). Every $\alpha \in$ Hom (S, T) is a homomorphism of S into G, hence $\alpha \in$ Hom (S, G). But then α has a unique extension $\bar{\alpha}$ to an endomorphism of G.

Next we show: When the elements of Hom (S, T) are identified with their extensions, then Hom (S, T) is a p-pure subgroup of Hom (G, G).

Let $\alpha \in$ Hom (S, T) and identify α with its extension in Hom (G, G). Suppose $\alpha = p^k \mu$, $\mu \in$ Hom (G, G). Then $p^k \mu(a) \in T$ for each $a \in S$. By p-purity of T in G, $\mu(a) \in T$ for each $a \in S$ and therefore $\mu \in$ Hom (S, T).

We have proved the well known

Theorem 4. *Let S and T be p-pure subgroups of a p-closed group G. Then each element of* Hom (S, T) *may be extended uniquely to an endomorphism of G and when the elements of* Hom (S, T) *are identified with their extensions, then* Hom (S, T) *is a p-pure subgroup of* Hom (G, G) (cf. [1], Lemma, p. 139).

Remark. In particular, if S is a p-pure subgroup of $G = \sum_{\lambda \in \Lambda}' Z(p)_\lambda$, then each $\alpha \in$ Hom (S, S) may be extended to an endomorphism $\bar{\alpha}$ of G and when we identify α and $\bar{\alpha}$, then Hom (S, S) is a p-pure subgroup of Hom $(G, G) \cong \sum_m^* G$, with $|\Lambda| = m$. The question now arises to characterize those p-pure subgroups S of p-closed groups G which have the additional property that Hom $(S, S) \cong \sum_m^* S$.

Let S be a p-pure subgroup of $G = \sum_{\lambda \in \Lambda}' Z(p)_\lambda$ containing $B = \sum_\lambda Z_\lambda$ as a p-basic subgroup. Both G and B are fixed and to avoid trivialities we suppose that $S \neq B$, $S \neq G$. In order that Hom $(S, S) \cong \sum_\lambda^* S$, we must have Hom $(B, S) \cong$ $\cong$ Hom (S, S), since Hom $(B, S) =$ Hom $(\sum_\lambda Z_\lambda, S) \cong \sum_\lambda^*$ Hom $(Z_\lambda, S) \cong \sum_\lambda^* S$. Therefore the groups S must have the property that every homomorphism of B into S can be extended to an endomorphism of S.

Now Hom $(S/B, S) = 0$, since S/B is p-divisible, but S is p-reduced. Since B is p-pure in S, we also have Ext $(S/B, G) = 0$, for G is torsion-free cotorsion (lemma 5).

As S/B is p-divisible and G is p-reduced, we get Hom $(S/B, G)=0$. Then

$$0 \to \mathrm{Hom}\,(S/B, S)=0 \to \mathrm{Hom}\,(S/B, G)=0 \to \mathrm{Hom}\,(S/B, G/S) \to \mathrm{Ext}\,(S/B, S) \to$$

$$\to \mathrm{Ext}\,(S/B, G)=0$$

is exact. Hence $\mathrm{Ext}\,(S/B, S) \cong \mathrm{Hom}\,(S/B, G/S)$. Likewise $0=\mathrm{Hom}\,(S/B, S) \to \to \mathrm{Hom}\,(S, S) \xrightarrow{\varphi} \mathrm{Hom}\,(B, S) \to \mathrm{Ext}\,(S/B, S) \to \mathrm{Ext}\,(S, S) \to \mathrm{Ext}\,(B, S)=0$ (B is free) is exact. If φ is *onto*, then we have $\mathrm{Ext}\,(S/B, S) \cong \mathrm{Ext}\,(S, S)$. In other words, if every $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension to an endomorphism of S, then $\mathrm{Ext}\,(S/B, S) \cong \cong \mathrm{Ext}\,(S, S)$. On the other hand, if $\mathrm{Ext}\,(S/B, S) \cong \mathrm{Hom}\,(S/B, G/S)=0$, then every $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension to an endomorphism of S. It can easily be shown that, if $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension $\bar{\alpha} \in \mathrm{Hom}\,(S, S)$, $\bar{\alpha}$ is uniquely determined. We remark also that $\mathrm{Ext}\,(S/B, S)=0$ always implies that $\mathrm{Ext}\,(S, S)=0$. $G = \sum_{\lambda \in \Lambda}' Z(p)_\lambda$ is q-divisible for all primes $q \neq p$, hence G/B, as a homomorphic image of G, is q-divisible for all primes $q \neq p$. But B is a p-basic subgroup of G, so G/B is p-divisible too. Hence *G/B is a divisible group.* Likewise $G/S \cong G/B/S/B$, as a homomorphic image of G/B, *is a divisible group.* Assume that G/S is not torsion-free, then the torsion-part $T \neq 0$ of G/S is a direct summand of G/S, hence $\mathrm{Ext}\,(G/S, G)=0$ implies $\mathrm{Ext}\,(T, G)=0$. It follows that the p-primary component of T is 0 by Lemma 5. So G/S cannot contain elements whose orders are powers of the prime p. In the same way we find that if S/B is a torsion-group, then $\mathrm{Ext}\,(S/B, G)=0$ implies that the p-primary component of S/B is 0. In particular S/B cannot be a p-group.

After these preliminary remarks we now investigate the groups S with

$$\mathrm{Hom}\,(S/B, G/S)=0.$$

We shall need the following

Lemma 6. *Let T be any group and suppose $T \subseteq D$, where D is a divisible group. Then* $\mathrm{Hom}\,(T, D/T)=0$ *implies that T is a divisible group.*

Proof. First we reduce the general case for arbitrary T to the case that T is torsion. If $D=T$, the lemma is trivial. So assume $D/T \neq 0$, and let T_t be the torsion subgroup of T. From $0 \to T_t \to T \to T/T_t \to 0$ is exact it follows that $0 \to \mathrm{Hom}\,(T/T_t, D/T) \to \to \mathrm{Hom}\,(T, D/T)$ is exact. But $\mathrm{Hom}\,(T, D/T)=0$, hence $\mathrm{Hom}\,(T/T_t, D/T)=0$. Suppose that $T/T_t \neq 0$. T/T_t is torsion-free, hence $Z \subseteq T/T_t$. Now $\mathrm{Hom}\,(T/T_t, D/T)=0 \to \to \mathrm{Hom}\,(Z, D/T) \to \mathrm{Ext}\,(T/T_t/Z, D/T)=0$ is exact so $\mathrm{Hom}\,(Z, D/T) \cong D/T=0$, which is a contradiction. Hence $T/T_t=0$ or $T=T_t$. From now on T is supposed to be a torsion group. If $T=0$, the lemma is trivial. Let $T \neq 0$, then T is the direct sum of its p-primary components and since $T \neq 0$, T contains an element of order p_i for some prime p_i. Then T contains a direct summand of type $C(p_i^l)$ ($l \geqq 1$ or $l=\infty$) ([6], p. 80).

Suppose $C(p_i^l)$ with finite $l \geqq 1$ is a direct summand of T, then Hom $(C(p_i^l), D/T)=0$. Since $C(p_i^l)$ is a direct summand of T, $C(p_i^\infty)/C(p_i^l)$ is a direct summand of D/T, hence Hom $(C(p_i^l), C(p_i^\infty))=0$. This gives a contradiction, since

$$\text{Hom}\,(C(p_i^l), C(p_i^\infty)) \cong C(p_i^l),$$

as is well known. So, if T contains an element of order p_i, T must contain a direct summand of type $C(p_i^\infty)$. Hence $T = E \oplus T'$, where E is the divisible part $(\neq 0)$ of T and T' is the reduced part of T. Suppose T' is not 0. As T' is torsion it contains an element of order p_j for some prime p_j. Then T' contains a direct summand of type $C(p_j^m)$, where $m \geqq 1$ (finite) or $m=\infty$. But T' is reduced, so it cannot contain a direct summand of type $C(p_j^\infty)$. Hence $C(p_j^m)$ is a direct summand of T' for $m \geqq 1$ and finite. But then $C(p_j^m)$ is a direct summand of T which is impossible as we have seen. Hence $T'=0$ and $T=E$ is divisible.

Next we prove:

Lemma 7. Hom $(S/B, G/S)=0$ *implies that* G/S *is not a torsion-group.*

Proof. $G/S \cong G/B/S/B$, so Hom $(S/B, G/S) \cong$ Hom $(S/B, G/B/S/B)=0$ implies that S/B is a divisible group by Lemma 6. Now Hom $(S/B, G/B/S/B)=0$ implies Hom $(S/B/(S/B)_t, G/B/S/B)=0$ (see the proof of Lemma 6) implies $S/B/(S/B)_t=0$ or $S/B=(S/B)_t$. It follows that S/B is a divisible torsion-group. Assume now that G/S is a torsion-group. Then S/B torsion and $G/B/S/B \cong G/S$ torsion imply G/B torsion which is a contradiction, since G/B contains $\sum_\lambda Z(p)_\lambda/\Sigma Z_\lambda \cong \Sigma Z(p)_\lambda/Z_\lambda$ as a direct summand and this is a mixed group. Consequently, G/S is not a torsion-group. This completes the proof of Lemma 7.

For the sake of reference we state the next lemma whose proof is contained in the proof of Lemma 7.

Lemma 8. Hom $(S/B, G/S)=0$ *implies that* S *and* B *have the same (torsion-free) rank and that* S/B *is divisible.*

Remark. By definitions 1. 5 and 1. 6 in [4], p.62 or the remark on p. 45 in [3], the torsion-free groups S with Hom $(S/B, G/S)=0$ are *quotient-divisible* groups, as S contains a free group B and S/B is a divisible torsion-group.

Now we prove:

Theorem 5. *Let* S *be a torsion-free group without elements of infinite p-height and with rank* $\leqq \aleph_0$. *Consider* S *as a p-pure subgroup of* $G = \sum_{\lambda \in \Lambda}' Z(p)_\lambda$, *while* S *contains* $B = \sum_{\lambda \in \Lambda} Z_\lambda$ *as a p-basic subgroup. Then the following are equivalent:*

(i) Ext $(S/B, S)=0$ *(or* Hom $(S/B, G/S)=0$*).*

(ii) *S and B have the same torsion-free rank and* $\mathrm{Ext}(S, S)=0$.

(iii) *There exists a torsion-free quotient-divisible group S' of rank 1 with $S' \subseteqq Q_p$, such that $S \cong \sum_{\lambda \in \Lambda} S'_\lambda$. ($Q_p$ is the group of all rationals with denominators prime to p).*

Proof. (i)→(ii) is clear from lemma 8 and the preliminary remarks of lemma 6.

(ii)→(iii). Since rank $S \leqq \varkappa_0$ we can apply lemma 4. 2 of J. HAUSEN in ([10], p. 170) which assures us of the existence of a group S' of rank 1, torsion-free and quotient-divisible, such that $S \cong \sum S'$ (direct sum). Since S' has rank 1 and S and B have the same rank (by (ii)) we must have $S \cong \sum_{\lambda \in \Lambda} S'_\lambda (|\Lambda| = \text{rank } S = \text{rank } B)$. Now S has no elements of infinite p-height, so S' (as a direct summand) has the same property. Then $S' \subseteqq Q_p$.

(iii)→(i). From $S \cong \sum_\lambda S', B = \sum_\lambda Z$ we infer that $S/B \cong \sum_\lambda S'/Z \cong \sum_\lambda \sum_{t \in P} C(t^\infty)$, where P is a set of primes and $p \notin P$, since $S' \subseteqq Q_p$. Now $\mathrm{Ext}(S/B, S) \cong$ $\cong \mathrm{Hom}(S/B, D/S)$, where D is the divisible hull of S ([6], p. 244). Since rank $S =$ $=$ rank $B = |\Lambda|$, we get $D = \sum_\lambda Q$. Hence $D/B \cong \sum_\lambda Q/Z \cong \sum_\lambda \sum_s C(s^\infty)$, where the summation $\sum_s$ is taken over all primes s. Then $D/S \cong D/B/S/B \cong$ $\cong \sum_\lambda (\sum_s C(s^\infty)/\sum_{t \in P} C(t^\infty)) \cong \sum_\lambda (\sum_{u \in C(P)} C(u^\infty))$, where $C(P)$ is the complement of P in the set of all primes. Then

$$\mathrm{Hom}(S/B, D/S) \cong \mathrm{Hom}\Big(\sum_\lambda \sum_{t \in P} C(t^\infty), \sum_\lambda \sum_{u \in C(P)} C(u^\infty)\Big) = 0.$$

This completes the proof of theorem 5.

It may be remarked that each of the conditions (i), (ii) and (iii) is sufficient in order that every $\alpha \in \mathrm{Hom}(B, S)$ may be extended uniquely to an endomorphism of S. Now we specialize to the case of finite rank. We recall that a non-nil group of rank 1 is a torsion-free group of rank 1 with characteristic $(k_1, k_2, \ldots, k_i, \ldots)$ with either $k_i = 0$ or $k_i = \infty$ for all i. The quotient-divisible groups of rank 1 are exactly the non-nil groups of rank 1.

Theorem 6. *Let n be a natural number $\geqq 1$. Let S be a torsion-free group and a proper p-pure subgroup of $G = \sum_1^n Z(p)$ while S contains $B = \sum_1^n Z$ as a p-basic subgroup. Then the following are equivalent:*

(i) $\mathrm{Hom}(S/B, G/S) = 0$ *(or* $\mathrm{Ext}(S/B, S) = 0$*).*

(ii) *S has rank n and* $\mathrm{Ext}(S, S) = 0$.

(iii) *S has rank n and every* $\alpha \in \mathrm{Hom}\,(B, S)$ *may be extended to an endomorphism of S.*

(iv) *S is isomorphic to the direct sum of n isomorphic non-nil groups S′ of rank* 1 *with* $S' \subseteqq Q_p$, *where* Q_p *is the group of all rationals with denominators prime to p.*

Clearly (i)$\leftrightarrow$(ii)$\leftrightarrow$(iv) by Theorem 5. That (iii)$\leftrightarrow$(iv) is a special case of the next result. We now investigate the p-pure subgroups S of $G = \sum_1^n Z(p)$ containing $B = \sum_1^n Z$ as a p-basic subgroup and with the property that every $\alpha \in \mathrm{Hom}\,(B, S)$ may be extended to an endomorphism of S. We do not assume that $\mathrm{Hom}\,(S/B, G/S)=0$.

Theorem 7. *Let S be a p-pure subgroup of* $G = \sum_1^n Z(p)$ *containing* $B = \sum_1^n Z$ *as a p-basic subgroup. Then the following are equivalent:*

(i) *Every* $\alpha \in \mathrm{Hom}\,(B, S)$ *may be extended to an endomorphism of S.*

(ii) *S is isomorphic to the direct sum of n isomorphic groups I, such that I is a subgroup of Z(p) which contains* 1 *and with the property* $\pi I \subseteqq I$ *for any* $\pi \in I$.

Proof. (i)$\to$(ii). Since S is p-pure in $G = \sum_1^n Z(p)$, every $\delta \in \mathrm{End}\, S$ has a unique extension $\bar{\delta} \in \mathrm{End}\, G$. So each element $\delta \in \mathrm{End}\, S$ is a (left) multiplication endomorphism by an $n \times n$-matrix with entries in $Z(p)$. Since $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots$ $\ldots, (0, 0, \ldots, 1) \in S$ the columns in the $n \times n$-matrix are elements of S. Now $\mathrm{Hom}\,(S, S) = \mathrm{Hom}\,(B, S) \cong \sum_1^n S$, so any $(\pi_1, \pi_2, \ldots, \pi_n) \in \sum_1^n S (\pi_i \in S)$ may be used as a multiplicator on the left, inducing an endomorphism of S, in other words,

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \cdots & \cdots & \cdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix} S \leqq S, \text{ whenever the columns are elements of S. Then}$$

$$\begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & 0 \ldots 0 \\ \cdots & \cdots \\ 0 & 0 \ldots 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_1 \\ \cdot \\ \pi_n \end{pmatrix} = \begin{pmatrix} \pi_1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \in S \quad \text{if} \quad \begin{pmatrix} \pi_1 \\ \pi_2 \\ \cdot \\ \pi_n \end{pmatrix} \in S.$$

Similar for other components. Hence in S we have the direct-sum decompositon:

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \cdot \\ \pi_n \end{pmatrix} = \begin{pmatrix} \pi_1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \pi_2 \\ \cdot \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \pi_n \end{pmatrix}.$$

The elements of the form $\begin{pmatrix} 0 \\ \cdot \\ \pi_j \\ 0 \end{pmatrix} \in S$ form the subgroup I_j in S. Then $S = I_1 \oplus \cdots \oplus I_n$.

If we identify $\begin{pmatrix} 0 \\ \cdot \\ \pi_j \\ 0 \end{pmatrix} \leftrightarrow \pi_j$, then each I_j is a subgroup of $Z(p)$. As a direct summand, I_j is a pure, hence p-pure, subgroup of S. S is p-pure in $G = \sum_1^n Z(p)$, so I_j is p-pure in $Z(p)$. Hence every map of I_j into I_k is the restriction of an endomorphism of $Z(p)$ (theorem 4), i.e. every map of I_j into I_k is a (left) multiplication by an element $\pi \in Z(p)$. Since $1 \in I_j$, $\pi.\ 1 = \pi \in I_k$. Then $\operatorname{Hom}(S, S) = \operatorname{Hom}(I_1 \oplus \cdots \oplus I_n, I_1 \oplus \cdots \oplus I_n) \cong$ $\cong \sum_{j,k} \operatorname{Hom}(I_j, I_k)$ and $\operatorname{Hom}(B, S) \cong \sum_1^n S = \sum_1^n (I_1 \oplus \cdots \oplus I_n)$ and every map in $\operatorname{Hom}(B, S)$ is the restriction of a map in $\operatorname{Hom}(S, S)$ imply $\operatorname{Hom}(I_j, I_k) \cong I_k$ $(j, k = 1, \ldots, n)$. Then $I_k I_j \subseteqq I_k$, but $\pi.\ 1 = \pi \in I_k$ for any $\pi \in I_k$ implies $I_k I_j = I_k$. Since $I_k I_j = I_j I_k$ it follows that $I_j = I_k$ $(j, k = 1, \ldots, n)$. So, if we put $I_j = I$, we get $S = I \oplus I \oplus$ $\oplus \cdots \oplus I$ (n summands). Moreover I is a subgroup of $Z(p)$ with $II = I$ or $\pi I \subseteqq I$ for any $\pi \in I$.

(ii) $\to$ (i). $S = \sum_1^n I$, where I is a subgroup of $Z(p)$ with $\pi I \subseteqq I$ for any $\pi \in I$. I is p-pure in S, S is p-pure in $\sum_1^n Z(p)$, so I is p-pure in $\sum_1^n Z(p)$, hence I is p-pure in $Z(p)$. So each $\alpha \in \operatorname{End} I$ has a unique extension to an endomorphism of $Z(p)$. Then each element of $\operatorname{End} I$ is a left multiplication endomorphism by an element $\pi \in Z(p)$. Since $1 \in I$, $\pi.\ 1 = \pi$ is in I. So $\operatorname{End} I \subseteqq L_I$, where L_I denotes the set of all left multiplication endomorphism by the elements of I. Then $\operatorname{End} I \subseteqq L_I$ and I, as a ring, is a subring of $Z(p)$ imply $\operatorname{End} I = L_I$ (Lemma, [1], p. 319), in other words $\operatorname{Hom}(I, I) \cong I$. From $\operatorname{Hom}(S, S) \cong \sum_1^{n^2} \operatorname{Hom}(I, I)$ and

$$\operatorname{Hom}(B, S) \cong \sum_1^{n^2} \operatorname{Hom}(Z, I)$$

we infer that every map of $\operatorname{Hom}(B, S)$ may be extended to an endomorphism of S. This completes the proof of Theorem 7.

Remark. If S has rank n, then (i) resp. (ii) of Theorem 7 pass into (iii) resp. (iv) of Theorem 6.

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(Received June 19, 1969)

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