Remarks on inequalities of series of positive terms

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We proved the following inequalities for non-negative a_n and b_n:

$$(1)\qquad \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} a_k^\gamma b_{n-k}^\gamma\right)^{1/\gamma} \geqq 1\left(\sum_{k=-\infty}^{\infty} a_k^r\right)^{1/r}\left(\sum_{k=-\infty}^{\infty} b_k^s\right)^{1/s}, \quad *)$$

where $1 \leqq r, s, \gamma \leqq \infty$ and $\frac{1}{r}+\frac{1}{s} = 1+\frac{1}{\gamma}$ (see Theorem in [1]); and

$$(2)\qquad (\sup_i a_i)(\sup_i b_i)+\sum_{n=-\infty}^{\infty} \sup_k a_k b_{n-k} \geqq p^{1/p}q^{1/q}\left(\sum_{k=-\infty}^{\infty} a_k^p\right)^{1/p}\left(\sum_{k=-\infty}^{\infty} b_k^q\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q} = 1$ (see Theorem 1 in [2]).

We also formulated in [2], without proof, the following inequality. If $\{a_n^{(i)}\}$ $(i=1, 2, \dots, m;\ n=0, \pm 1, \pm 2, \dots)$ are m sequences of non-negative numbers, then

$$(3)\qquad (m-1)\prod_{i=1}^{m}(\sup_k a_k^{(i)})+\sum_{n=-\infty}^{\infty}\ \sup_{k_1+k_2+\dots+k_m=n} a_{k_1}^{(1)} a_{k_2}^{(2)}\cdots a_{k_m}^{(m)} \geqq$$
$$\geqq \prod_{i=1}^{m}(p_i)^{1/p_i}\prod_{i=1}^{m}\left(\sum_{k=-\infty}^{\infty}(a_k^{(i)})^{p_i}\right)^{1/p_i},$$

where $1 \leqq p_i \leqq \infty$ and $\sum_{i=1}^{m}\frac{1}{p_i}=1$.

In the present note *we show that the constant factors* 1, $p^{\frac{1}{p}} q^{\frac{1}{q}}$, *and* $\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}$ *in* (1), (2), *and* (3), *respectively, are best possible.*

Since inequality (2) is a special case of (3), it would be sufficient to prove that the constant factor in (3) is best possible. In spite of this we prove both cases, because the idea of proof can be seen much better in the simple case, furthermore the proof of the general case is not a trivial straightforward generalization.

*) If $\gamma=\infty$ and $C_k \geqq 0$, then $\left\{\sum_{n=-\infty}^{\infty} C_n^\gamma\right\}^{1/\gamma}$ means $\sup_n C_n$.

Moreover, with respect to the obvious inequality

$$\left(\sum_{k=-\infty}^{\infty} a_k^\gamma b_{n-k}^\gamma\right)^{1/\gamma} \geqq \sup_k a_k b_{n-k},$$

which holds for any positive γ and n, we remark that (2) implies the following

Corollary. *Suppose that $\gamma>0$, $1\leqq p, q\leqq\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$(4)\qquad (\sup_n a_n)(\sup_n b_n)+\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} a_k^\gamma b_{n-k}^\gamma\right)^{1/\gamma} \geqq$$
$$\geqq p^{1/p}q^{1/q}\left(\sum_{n=-\infty}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=-\infty}^{\infty} b_n^q\right)^{1/q}.$$

The factor $p^{\frac{1}{p}}q^{\frac{1}{q}}$ in (4) seems not to be best possible generally. But if $p=q=2$ then it is best possible for any $\gamma>0$. Namely, if $a_0=b_0=1$ and $a_n=b_n=0$ if $n\neq 0$, then both sides of (4) equal 2.

Setting the above given sequences $\{a_n\}$ and $\{b_n\}$ into (1) we obtain an equality. This verifies that the factor 1 in (1) is best possible.

The proof of the fact that the factor $p^{\frac{1}{p}}q^{\frac{1}{q}}$ in (2) is best possible is a little bit longer.

It is easy to see that if $a_0=a_1=\dots=a_{\nu-1}=1$, $b_0=b_1=\dots=b_{\mu-1}=1$ and $a_n=b_n=0$ otherwise, then inequality (2) reduces to

$$(5)\qquad \nu+\mu\geqq p^{1/p}q^{1/q}\nu^{1/p}\mu^{1/q}.$$

If we can show that for any positive ε there exist integers ν and μ such that

$$(6)\qquad \nu+\mu<(p^{1/p}q^{1/q}+\varepsilon)\nu^{1/p}\mu^{1/q},$$

then our statement will be proved.

Inequality (6) is equivalent to

$$(7)\qquad \left(\frac{\nu}{\mu}\right)^{1/q}+\left(\frac{\mu}{\nu}\right)^{1/p}<p^{1/p}q^{1/q}+\varepsilon\quad\left(\frac{1}{p}+\frac{1}{q}=1\right).$$

If $p=1$ and $q=\infty$ then (7) means

$$1+\frac{\mu}{\nu}<1+\varepsilon,$$

and this obviously follows if $\mu=1$ and ν is large enough.

The case $q=1$ and $p=\infty$ can be verified similarly.

If $1<p, q<\infty$ then we have

$$(8)\qquad \left(\frac{q}{p}\right)^{1/q}+\left(\frac{p}{q}\right)^{1/p}=p^{1/p}q^{1/q}\left(\frac{1}{p^{1/p+1/q}}+\frac{1}{q^{1/p+1/q}}\right)=p^{1/p}q^{1/q}.$$

Since the functions $y=x^\alpha$ $(\alpha>0)$ are continuous at any point $x_0>0$, by (8) we have that

$$x^{1/q}+\left(\frac{1}{x}\right)^{1/p}\to p^{1/p}q^{1/q} \text{ as } x\to\frac{q}{p}.$$

Consequently if $x=\frac{\nu}{\mu}$ approximates the value $\frac{q}{p}$ "in a suitable way", then (7) and (6) hold. This proves that the factor $p^{\frac{1}{p}}q^{\frac{1}{q}}$ in (2) is best possible.

To prove that the factor $\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}$ in (3) is best possible, we start with the case $p_1=1$ and $p_2=\dots=p_m=\infty$.

Set $a_0^{(i)}=a_1^{(i)}=\dots=a_{\nu_i-1}^{(i)}=1$ and $a_k^{(i)}=0$ otherwise, for $i=1,2,\dots,m$. Then inequality (3) reduces to

$$m-1+1+\sum_{i=1}^{m}(\nu_i-1)\geqq 1\cdot\nu_1.$$

Hence, if $\nu_2=\nu_3=\dots=\nu_m=1$ and ν_1 is large enough, it can be seen that $1\left(=\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}\right)$ is best possible indeed.

If $1<p_1,p_2,\dots,p_\mu<\infty$ and $p_{\mu+1}=\dots=p_m=\infty$, then inequality (3) reduces to

$$(m-1)+1+\sum_{i=1}^{m}(\nu_i-1)\geqq\prod_{i=1}^{\mu}(p_i)^{1/p_i}\prod_{i=1}^{\mu}(\nu_i)^{1/p_i}. \tag{9}$$

Since $\sum_{i=1}^{m}\frac{1}{p_i}=\sum_{i=1}^{\mu}\frac{1}{p_i}=1$ we have

$$\sum_{j=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{p_i}{p_j}\right)^{1/p_i}=\prod_{i=1}^{\mu}(p_i)^{1/p_i}\sum_{j=1}^{\mu}\frac{1}{p_j^{\sum_{i=1}^{\mu}1/p_i}}=\prod_{i=1}^{\mu}(p_i)^{1/p_i}. \tag{10}$$

By (10) it can be seen that for any $\varepsilon>0$ there exist rational numbers r_i $(i=1,\dots,\mu)$ such that

$$\sum_{j=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{r_i}{r_j}\right)^{1/p_i}<\prod_{i=1}^{\mu}(p_i)^{1/p_i}+\varepsilon. \tag{11}$$

Choose the positive integers $l_1,\cdots,l_\mu$ such that

$$\frac{r_1}{r_2}=\frac{l_2}{l_1},\quad\frac{r_1}{r_3}=\frac{l_3}{l_1},\ \cdots,\ \frac{r_1}{r_\mu}=\frac{l_\mu}{l_1},$$

that is,

$$r_1l_1=r_2l_2=r_3l_3=\dots=r_\mu l_\mu. \tag{12}$$

Define $v_1=l_1N$, $v_2=l_2N$, ..., $v_\mu=l_\mu N$ and $v_{\mu+1}=\cdots=v_m=1$, where N is a natural number to be defined later. Then inequality (9) has the form

$$m+\sum_{i=1}^{\mu}(Nl_i-1)\geqq\prod_{i=1}^{\mu}(p_i)^{1/p_i}\prod_{i=1}^{\mu}(Nl_i)^{1/p_i}.$$

We show that if N is large enough then

$$m+\sum_{i=1}^{\mu}(Nl_i-1)<\left(\prod_{i=1}^{\mu}(p_i)^{1/p_i}+\varepsilon\right)\prod_{i=1}^{\mu}(Nl_i)^{1/p_i}, \tag{13}$$

which verifies that the factor $\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}$ is best possible. By (12),

$$\frac{m-\mu+N\sum_{j=1}^{\mu}l_j}{N\prod_{i=1}^{\mu}(l_i)^{1/p_i}}<\frac{m-\mu}{N}+\sum_{j=1}^{\mu}\frac{l_j^{\sum_{i=1}^{\mu}1/p_i}}{\prod_{i=1}^{\mu}(l_i)^{1/p_i}}=\frac{m-\mu}{N}+\sum_{i=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{l_j}{l_i}\right)^{1/p_i}=$$

$$=\frac{m-\mu}{N}+\sum_{j=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{r_i}{r_j}\right)^{1/p_i}.$$

Hence, if N is large enough we obtain by (11) that

$$\frac{m-\mu+N\sum_{j=1}^{\mu}l_j}{N\prod_{i=1}^{\mu}(l_i)^{1/p_i}}<\prod_{i=1}^{\mu}(p_i)^{1/p_i}+\varepsilon.$$

So we have proved (13), and this completes our proof.

References

[1] L. Leindler, On some inequalities concerning series of positive terms, *Acta Sci. Math.*, **33** (1972), 11—14.

[2] L. Leindler, On some inequalities of series of positive terms, *Math. Z.*, **128** (1972), 305—309.

(Received November 9, 1972)