Function algebras and flows

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§ 1. Throughout this article X will denote a fixed compact Hausdorff space upon which the real line $\mathbf{R}$ (with the usual topology) acts as a locally compact transformation group. The pair $(X, \mathbf{R})$ will be called a *flow* and the translate of an x in X by a t in $\mathbf{R}$ will be written $x+t$. The space of all continuous complex-valued functions on X will be denoted by $C(X)$. If φ is a function in $C(X)$, then φ will be called *analytic* in case for each x in X the function $\varphi(x+t)$ of t is the boundary function of a function which is bounded and analytic in the upper half-plane. The space of all analytic functions in $C(X)$ will be denoted by $\mathfrak{A}$. It is clear that $\mathfrak{A}$ is a uniformly closed subalgebra of $C(X)$ which contains the constant functions.

This notion of analyticity was first defined by FORELLI and has been studied extensively by him in a number of articles (see [1], [2], [3], [4] and [5]). Our principal objective in this article is to show that under suitable conditions the algebra $\mathfrak{A}$ belongs to well known classes of abstract function algebras.

Recall that if $\mathfrak{B}$ is an algebra of continuous functions on a compact Hausdorff space Y then a probability measure m on Y is called a *representing measure* for $\mathfrak{B}$ in case $\int_Y \varphi\psi\, dm = \left(\int_Y \varphi\, dm\right)\left(\int_Y \psi\, dm\right)$ for all φ and ψ in $\mathfrak{B}$. If m is a representing measure for $\mathfrak{B}$ and if $\mathfrak{B}$ contains the constant functions, then $\mathfrak{B}$ is called a *weak-$*$ Dirichlet algebra* in $L^\infty(m)$ in case $\mathfrak{B}+\overline{\mathfrak{B}}$ is weak-$*$ dense in $L^\infty(m)$. (The bar denotes conjugation, here and always.) We refer the reader to [16] for an account of weak-$*$ Dirichlet algebras. If $\mathfrak{B}$ contains the constant functions and if $\mathfrak{B}+\overline{\mathfrak{B}}$ is uniformly dense in the space of continuous functions on Y, then $\mathfrak{B}$ is called a *Dirichlet algebra*.

Our first basic structure theorem is

Theorem I. *If μ is an invariant, ergodic, probability measure on X, then μ is a representing measure for $\mathfrak{A}$ and $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(\mu)$.*

With an additional hypothesis on the flow $(X, \mathbf{R})$ er are able to prove a much stronger theorem. The hypothesis is that $(X, \mathbf{R})$ is *strictly* (or *uniquely*) *ergodic*

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in the sense that there is exactly one probability measure on X which is invariant under the action of $\mathbf{R}$. It is well known that since X is compact there is at least one invariant probability measure on X. However, the requirement that there is exactly one is very special (see [6], [11], and [14]). Note that if $(X, \mathbf{R})$ is strictly ergodic, then the unique invariant probability measure must be ergodic and this explains the terminology.

Our second basic structure theorem is

Theorem II. *If the flow* $(X, \mathbf{R})$ *is strictly ergodic, then* $\mathfrak{A}$ *is a Dirichlet algebra on* X.

Theorems I and II enable us to exhibit new and striking ways in which algebras of analytic functions associated with flows generalize certain spaces of generalized analytic functions in the sense of Arens and Singer. To see this recall how these spaces are defined. Let Γ be a non-zero subgroup of $\mathbf{R}$, give Γ the discrete topology, and let G be its compact character group. The space $\mathfrak{A}_\Gamma$ of generalized analytic functions determined by Γ is defined to be the space of all continuous complex-valued functions on G whose Fourier transforms vanish on the negative half of Γ. It is easy to see that $\mathfrak{A}_\Gamma$ is a Dirichlet algebra on G. If Γ is (isomorphic to) the integers, then $\mathfrak{A}_\Gamma$ is simply the classic disc algebra. In general $\mathfrak{A}_\Gamma$ may be regarded as the algebra of analytic almost periodic functions on the line whose frequencies lie in Γ. The real line can be imbedded in G as the space of all characters on Γ which are continuous with respect to the usual topology of $\mathbf{R}$ restricted to Γ. The imbedding defines a natural action of $\mathbf{R}$ on G so that $(G, \mathbf{R})$ is a flow. It is not hard to see that the space of analytic functions associated with this flow is precisely $\mathfrak{A}_\Gamma$. The flow $(G, \mathbf{R})$ is strictly ergodic — normalized Haar measure on G is the unique probability measure on G which is invariant under the action of $\mathbf{R}$. Thus Theorem II gives a new, albeit roundabout, proof that $\mathfrak{A}_\Gamma$ is a Dirichlet algebra. There are many flows, even strictly ergodic ones, which are not of the form $(G, \mathbf{R})$ and consequently, the results which we obtain for spaces of analytic functions associated with general flows represent bona fide extensions of results known to hold for spaces of generalized analytic functions.

Section 2 is devoted to the proofs of Theorems I and II. In section 3 we investigate the nature of general representing measures for $\mathfrak{A}$. Our investigation is incomplete in some respects; however, it is sufficiently complete to yield results which are complementary to Forelli's generalization of the F. and M. Riesz Theorem [1]. Specifically, we present conditions on a representing measure m under which the abstract Hardy spaces $H^p(m)$, $1 \leq p \leq \infty$, have the property that no non-zero function in $H^p(m)$ can vanish on a set of positive measure. These conditions are satisfied in the following two situations:

(i) $(X, \mathbf{R})$ is an arbitrary flow and m is invariant and ergodic.

(ii) $(X, \mathbf{R})$ is strictly ergodic, and m is an arbitrary representing measure for $\mathfrak{A}$. This feature of the Hardy spaces is well known when $\mathfrak{A}$ is the disc algebra. In [9] HELSON and LOWDENSLAGER showed that if $\mathfrak{A}_\Gamma$ is the algebra of generalized analytic functions associated with the subgroup Γ of $\mathbf{R}$, and if m is Haar measure on G then the Hardy spaces $H^p(m)$, $1 \leqq p \leqq \infty$, also have this feature. In the situations (i) and (ii), we also show that $H^\infty(m)$ is a maximal weak-$*$ closed subalgebra of $L^\infty(m)$. This result is complementary to FORELLI's generalization [5] of WERMER's maximality theorem.

§ 2. The dual space of $C(X)$ is the space of all bounded, complex, Baire measures on X; we will denote it by $M(X)$. If φ is in $C(X)$ and if λ is in $M(X)$, the integral $\int_X \varphi \, d\lambda$ will often be written as $\langle \varphi, \lambda \rangle$.

The definition of analyticity given in section 1, while intuitively appealing, is not the one which we shall use in our proofs. We digress momentarily in order to give an equivalent definition.

The action of $\mathbf{R}$ on X induces a strongly continuous, one-parameter group $\{T_t\}_{t \in \mathbf{R}}$ of automorphisms of $C(X)$. These are defined by the formula

$$(T_t \varphi)(x) = \varphi(x-t), \quad \varphi \in C(X), \quad t \in \mathbf{R}.$$

Using $\{T_t\}_{t \in \mathbf{R}}$ one may convolve a function in $C(X)$ or a measure in $M(X)$ with a function in $L^1(\mathbf{R})$ in the following way. For φ in $C(X)$ and f in $L^1(\mathbf{R})$, $\varphi * f$ is defined by setting

$$\varphi * f = \int_{-\infty}^{\infty} (T_t \varphi) f(t) \, dt.$$

If λ is in $M(X)$ and if f is in $L^1(\mathbf{R})$, $\lambda * f$ is defined to be the measure such that

$$\langle \varphi, \lambda * f \rangle = \langle \varphi * \tilde{f}, \lambda \rangle$$

for all φ in $C(X)$ where $\tilde{f}$ is the function whose value at t in $\mathbf{R}$ is $f(-t)$. Under these operations of convolution $C(X)$ and $M(X)$ become $L^1(\mathbf{R})$-modules such that

$$\|\varphi * f\| \leqq \|\varphi\| \, \|f\| \quad \text{and} \quad \|\lambda * f\| \leqq \|\lambda\| \, \|f\|$$

for all φ in $C(X)$, λ in $M(X)$, and f in $L^1(\mathbf{R})$. For each φ in $C(X)$ (resp., λ in $M(X)$) let $J(\varphi)$ (resp., $J(\lambda)$) be $\{f \in L^1(\mathbf{R}) | \varphi * f = 0\}$ (resp., $\{f \in L^1(\mathbf{R}) | \lambda * f = 0\}$). Then $J(\varphi)$ (resp., $J(\lambda)$) is an ideal in $L^1(\mathbf{R})$ which is closed by the above inequalities. The hull of $J(\varphi)$ (resp., $J(\lambda)$) is defined to be the *spectrum* of φ (resp., λ) in the sense of spectral synthesis and will be denoted by $\operatorname{sp}(\varphi)$ (resp., $\operatorname{sp}(\lambda)$). (Recall that the hull of an ideal in $L^1(\mathbf{R})$ is by definition the intersection of the zero sets of the Fourier transforms of the functions in the ideal.) We refer the reader to [1] for an account of spectra.

The equivalent formulation of analyticity is given in Proposition 2. 1 below. It was used by FORELLI in [5] although he never formally stated or proved it. The proof is an easy calculation based on the well known fact that a bounded measurable function F on $\mathbf{R}$ is the boundary function of a function which is analytic and bounded in the upper half-plane if and only if the spectrum of F in the usual sense of spectral synthesis is non-negative. Because of this, the proof will not be given.

Proposition 2. 1. *A function φ in $C(X)$ is analytic if and only if* $\operatorname{sp}(\varphi)\subseteqq[0, \infty)$.

Following FORELLI [1] we shall call a measure in $M(X)$ *analytic* in case its spectrum is non-negative.

The proofs of Theorems I and II rest on

Proposition 2. 2. *A measure λ in $M(X)$ is invariant under the action of* $\mathbf{R}$ *if and only if* $\operatorname{sp}(\lambda)$ *is contained in the singleton* $\{0\}$.

Proof. Suppose λ is invariant. If φ is in $C(X)$ and if f is in $L^1(\mathbf{R})$, then by Fubini's Theorem we obtain the equation

$$\langle\varphi, \lambda*f\rangle=\langle\varphi*\hat{f}, \lambda\rangle=\int_{-\infty}^{\infty}\langle T_t\varphi, \lambda\rangle\hat{f}(t)\,dt=\langle\varphi, \lambda\rangle\int_{-\infty}^{\infty}f(t)\,dt=\langle\varphi, \lambda\rangle\hat{f}(0)$$

where $\hat{f}$ is the Fourier transform of f. It follows easily that $\operatorname{sp}(\lambda)\subseteqq\{0\}$.

Suppose, conversely, that $\operatorname{sp}(\lambda)\subseteqq\{0\}$. Choose φ in $C(X)$ and for t in $\mathbf{R}$ let $F(t)=\langle T_t\varphi, \lambda\rangle$. On page 50 of [1] FORELLI showed that the spectrum of F as a bounded continuous function is contained in $-\operatorname{sp}(\varphi)\cap\operatorname{sp}(\lambda)\subseteqq\{0\}$. By [15, 7. 8. 3 (e)], F is constant. Thus for all φ in $C(X)$ and all t in $\mathbf{R}$, $\langle T_t\varphi, \lambda\rangle=\langle\varphi, \lambda\rangle$, and consequently, λ is invariant.

Proof of Theorem I:

(i) Let $\mathfrak{A}_\mu=\{\varphi\in\mathfrak{A}|\langle\varphi, \mu\rangle=0\}$. To show that μ is a representing measure for $\mathfrak{A}$ it clearly suffices to show that $\mathfrak{A}_\mu$ is an ideal in $\mathfrak{A}$. To this end, let $\mathfrak{A}_0$ be the intersection of $\mathfrak{A}$ with the weak-$*$ closure in $L^\infty(\mu)$ of the space of all functions φ in $\mathfrak{A}$ such that $\operatorname{sp}(\varphi)\subseteqq(0, \infty)$. Then by Lemma 3 and Theorem 1 of [1] it is easy to see that $\mathfrak{A}_0$ is a closed ideal in $\mathfrak{A}$. We show that $\mathfrak{A}_\mu=\mathfrak{A}_0$. Since μ is invariant, $\operatorname{sp}(\mu)\subseteqq\{0\}$ by Proposition 2. 2, and so μ is analytic. Hence, by Proposition 2 of [1], $\mathfrak{A}_0\subseteqq\mathfrak{A}_\mu$. Suppose φ is a function in $\mathfrak{A}_\mu$ which is not in $\mathfrak{A}_0$. Then by the Hahn—Banach Theorem there is a function f in $L^1(\mu)$ such that $\langle\varphi, f\,d\mu\rangle=1$ while $\langle\psi, f\,d\mu\rangle=0$ for all ψ in $\mathfrak{A}_0$. Observe that since $f\,d\mu$ annihilates $\mathfrak{A}_0$, $f\,d\mu$ is an analytic measure by Proposition 2 of [1]. For t in $\mathbf{R}$, let $F(t)=\langle T_t\varphi, f\,d\mu\rangle$. Then as in the proof of Proposition 2. 2, we find that $\operatorname{sp}(F)\subseteqq-\operatorname{sp}(\varphi)\cap\operatorname{sp}(f\,d\mu)\subseteqq(-\infty, 0)\cap[0, \infty)=\{0\}$, and so F is constant. The constant is $\langle\varphi, f\,d\mu\rangle=1$. For positive τ, let

$\varphi_\tau = \frac{1}{2\tau} \int_{-\tau}^{\tau} (T_t\varphi)\,dt$. Then by the individual ergodic theorem [8, p. 18] and the fact that μ is ergodic, the φ_τ converge a.e. (μ) to $\langle\varphi, \mu\rangle = 0$. The convergence of the φ_τ is also bounded and so we find that

$$\lim_{\tau\to\infty} \langle \varphi_\tau, f\,d\mu\rangle = 0.$$

On the other hand, by Fubini's Theorem,

$$\langle \varphi_\tau, f\,d\mu\rangle = \frac{1}{2\tau}\int_{-\tau}^{\tau} \langle T_t\varphi, f\,d\mu\rangle = \frac{1}{2\tau}\int_{-\tau}^{\tau} F(t)\,dt = 1$$

for all positive τ. Thus

$$\lim_{\tau\to\infty} \langle \varphi_\tau, f\,d\mu\rangle = 1.$$

This contradiction shows that $\mathfrak{A}_\mu = \mathfrak{A}_0$ as we promised and the proof of the first half of Theorem I is complete.

(ii) To show that $\mathfrak{A} + \overline{\mathfrak{A}}$ is weak-$*$ dense in $L^\infty(\mu)$ suppose f is a function in $L^1(\mu)$ which annihilates $\mathfrak{A} + \overline{\mathfrak{A}}$. By Proposition 2′ of [1] and the fact that

$$\overline{\mathfrak{A}} = \{\varphi \in C(X) | \mathrm{sp}\,(\varphi) \subseteqq (-\infty, 0]\}$$

[1, p. 48], we find that $\mathrm{sp}\,(f d\mu) \subseteqq \{0\}$. By Proposition 2. 2, $f d\mu$ is an invariant measure; and since μ is ergodic by hypothesis, f is constant. Since, however, the measure $f d\mu$ annihilates the constants, $f=0$. Whence $\mathfrak{A} + \overline{\mathfrak{A}}$ is weak-$*$ dense in $L^\infty(\mu)$ and the proof of Theorem I is complete.

If λ is in $M(X)$ and if t is in $\mathbf{R}$, we also write $T_t\lambda$ for the measure whose value at a Baire set E is $\lambda(E+t)$. The total variation measure of a measure λ in $M(X)$ will be denoted by $|\lambda|$. Recall that an arbitrary measure λ in $M(X)$ is called quasi-invariant in case $T_t|\lambda|$ and $|\lambda|$ have the same null sets for each t in $\mathbf{R}$.

The proof of the following lemma is a straightforward application of the definition of the term total variation measure and so will not be given.

Lemma 2. 3. *Let λ be a measure in $M(X)$ which is invariant under the action of $\mathbf{R}$ on X. Then $|\lambda|$ also is an invariant measure and the Radon—Nikodym derivative $\frac{d\lambda}{d|\lambda|}$ is (after modification on a $|\lambda|$-null set) an invariant function on X.*

Proof of Theorem II:

The unique probability measure on X which is invariant under the action of $\mathbf{R}$ will be denoted by μ. Recall that μ is necessarily ergodic.

We must show that $\mathfrak{A}+\overline{\mathfrak{A}}$ is uniformly dense in $C(X)$. Suppose the contrary and let λ be a measure in $M(X)$ of unit norm which annihilates $\mathfrak{A}+\overline{\mathfrak{A}}$. Then by Proposition 2′ of [1] and the fact that $\overline{\mathfrak{A}}=\{\varphi\in C(X)|\text{sp}(\varphi)\subseteqq(-\infty, 0]\}$ [1, p. 48], we see that $\text{sp}(\lambda)\subseteqq\{0\}$. By Proposition 2. 2, λ is invariant and so, by Lemma 2. 3, $|\lambda|$ is invariant. Since λ has norm one, $|\lambda|$ is a probability measure, and by the strict ergodicity of $(X, \mathbf{R})$, $|\lambda|=\mu$. Since by Lemma 2. 3, $d\lambda/d|\lambda|$ is invariant and since $|\lambda|=\mu$ is ergodic, $d\lambda/d|\lambda|$ is constant. Thus λ is a constant multiple of μ. But the multiple must be zero because λ annihilates the constant functions, and thus we have arrived at a contradiction. Whence $\mathfrak{A}+\overline{\mathfrak{A}}$ is uniformly dense in $C(X)$ and the proof of Theorem II is complete.

§ 3. In this section we investigate the properties of representing measures for $\mathfrak{A}$ on X. We note in advance that we will use the following fact several times in our arguments. If m is an arbitrary representing measure for $\mathfrak{A}$ on X, then each real-valued function in $H^2(m)$ is constant (see [7, p. 98]).

For each t in $\mathbf{R}$ we let C_t denote the closure in $C(X)$ of the space of functions φ in $C(X)$ such that $\text{sp}(\varphi)\subseteqq(t, \infty)$. By Lemma 3 of [1], C_t is a linear subspace of $C(X)$; and if $t\geqq 0$, C_t is an ideal in $\mathfrak{A}$ by Theorem 1 of [1].

If $\{M_\alpha\}_{\alpha\in A}$ is a family of closed subspaces of a Hilbert space then $\bigvee_{\alpha\in A} M_\alpha$ will denote their span and $\bigwedge_{\alpha\in A} M_\alpha$ will denote their intersection. Similarly, if $\{P_\alpha\}_{\alpha\in A}$ is a family of orthogonal projections on a Hilbert space, then $\bigvee_{\alpha\in A} P_\alpha$ will denote their least upper bound and $\bigwedge_{\alpha\in A} P_\alpha$ will denote their greatest lower bound.

If μ is a positive Baire measure on X, then we will often regard functions in $L^\infty(\mu)$ as multiplication operators on $L^2(\mu)$. Note that when $L^\infty(\mu)$ is regarded as an algebra of operators on $L^2(\mu)$, it is the closure of $C(X)$ in the weak operator topology; and observe that the subspaces of $L^2(\mu)$ which reduce $C(X)$ are of the form $\chi_E L^2(\mu)$ where χ_E denotes the characteristic function of the Baire set E.

Theorem III. *If m is a representing measure for $\mathfrak{A}$ on X which is not a point mass, then m is quasi-invariant under the action of* $\mathbf{R}$.

Proof. The proof rests on Theorem 2 of [1]. For each t in $\mathbf{R}$, let M_t be the closure of C_t in $L^2(m)$. The family $\{M_t\}_{t\in\mathbf{R}}$ is a decreasing family of subspaces of $L^2(m)$. Since the space of continuous functions on X with compact spectrum is dense in $C(X)$ [1, Lemma 3] and is contained in $\bigvee_{t\in\mathbf{R}} M_t$, it follows that $\bigvee_{t\in\mathbf{R}} M_t= =L^2(m)$. If φ is in $C(X)$ with $\text{sp}(\varphi)\subseteqq(s, \infty)$, then $\varphi C_t\subseteqq C_{t+s}$ for all t by [1, Theorem 1] and so $\varphi M_t\subseteqq M_{t+s}$ for all t. From this it follows that $\bigwedge_{t\in\mathbf{R}} M_t$ is reduced by each continuous function with compact spectrum. Thus $\bigwedge_{t\in\mathbf{R}} M_t$ is reduced by $C(X)$, and, by the above remarks must be of the form $\chi_E L^2(m)$ for some Baire set E. Since $\chi_E L^2(m) = \bigwedge_{t\in\mathbf{R}} M_t \subseteqq H^2(m)$, the facts that $H^2(m)$ contains no non-constant real-valued functions and that m is not a point mass allow us to conclude that $\bigwedge_{t\in\mathbf{R}} M_t$ is the zero space. Thus we have shown that $\{M_t\}_{t\in\mathbf{R}}$ satisfies the hypotheses of part 3 in Theorem 2 of [1]. Whence, by that theorem, m is quasi-invariant, and the proof of Theorem III is complete.

Theorem IV. *If m is a representing measure for $\mathfrak{A}$ on X such that $H^2(m)$ contains functions other than constants, then m is ergodic if and only if no non-zero function in $H^2(m)$ vanishes on a set of positive measure.*

Proof. The hypothesis implies that m is not a point mass so that by Theorem III m is quasi-invariant.

Suppose m is ergodic and let f be a function in $H^2(m)$ which vanishes on a set of positive measure. If E is the support of f, then $m(E)<1$ and we must show that $m(E)=0$. Observe that the smallest subspace of $L^2(m)$ which contains f and reduces $C(X)$ is $\chi_E L^2(m)$. Let $d\mu=\chi_E dm$, identify $L^2(\mu)$ with $\chi_E L^2(m)$, and for each t in $\mathbf{R}$ let $\mathscr{K}_t$ be the closed linear span in $L^2(\mu)$ of the space $\{\varphi f | \varphi\in C_t\}$. It follows easily from the proof of Theorem III that $\bigvee_{t\in\mathbf{R}} \mathscr{K}_t=L^2(\mu)$ and that $\bigwedge_{t\in\mathbf{R}} \mathscr{K}_t=\chi_F L^2(\mu)$ for some Baire set F. Since $\bigwedge_{t\in\mathbf{R}} \mathscr{K}_t\subseteqq H^2(m)$, $m(F)=0$ as before; so $\bigwedge_{t\in\mathbf{R}} \mathscr{K}_t$ is the zero subspace of $L^2(\mu)$. Finally, since the family $\{\mathscr{K}_t\}_{t\in\mathbf{R}}$ is decreasing and since $\varphi\mathscr{K}_t\subseteqq \subseteqq\mathscr{K}_{t+s}$ for all φ in $C(X)$ with sp $(\varphi)\subseteqq(s,\infty)$ we may apply Theorem 2 of [1] again to conclude that μ is quasi-invariant. However, m is quasi-invariant and is also ergodic by hypothesis. Therefore, since μ is absolutely continuous with respect to m, μ must be the zero measure by Lemma 9 of [1]. Thus $m(E)=0$ as was to be shown.

To prove the converse, assume m is not ergodic and let E be an invariant Baire set such that $0<m(E)<1$. We will produce a non-zero function in $H^2(m)$ which is supported either on E or on $X-E$.

For each t in $\mathbf{R}$, let M_t be the closure of C_t in $L^2(m)$ and let P_t be the projection of $L^2(m)$ onto M_t. It was shown in the proof of Theorem III that since m is not a point mass on X, $\{M_t\}_{t\in\mathbf{R}}$ is a decreasing family of subspaces of $L^2(m)$ whose span is $L^2(m)$ and whose intersection is the zero subspace. It is also easy to see that for each t in $\mathbf{R}$, $M_t = \bigvee_{s>t} M_s$. Thus, except for orientation, the family $\{P_t\}_{t\in\mathbf{R}}$ is a resolution of the identity which is continuous from the right, i.e., $\{P_t\}_{t\in\mathbf{R}}$ has the following four properties: (i) $\bigvee_{t\in\mathbf{R}} P_t$ is the identity operator on $L^2(m)$; (ii) $\bigwedge_{t\in\mathbf{R}} P_t$ is the zero operator; (iii) if $t<s$, then $P_s\leqq P_t$; and (iv) for each t in $\mathbf{R}$, $P_t = \bigvee_{s>t} P_s$.

Let $\{V_t\}_{t\in\mathbf{R}}$ be the strongly continuous unitary representation of $\mathbf{R}$ on $L^2(m)$ defined by the formula

$$V_t = \int_{-\infty}^{\infty} e^{-its}\, dP_s, \quad t \in \mathbf{R}.$$

Then, as Forelli showed in the proof of Theorem 2 in [1],

$$V_t \varphi V_t^* = T_t \varphi \tag{3. 1}$$

for all φ in $C(X)$ and all t in $\mathbf{R}$. Since $L^\infty(m)$ is the closure of $C(X)$ in the weak operator topology, equation (3. 1) is valid for all functions in $L^\infty(m)$ provided, of course, that the right hand side of the equation is interpreted in the obvious way. Since E is invariant by assumption, equation (3. 1) implies that χ_E commutes with $\{V_t\}_{t\in\mathbf{R}}$. Whence χ_E commutes with $\{P_t\}_{t\in\mathbf{R}}$ and so leaves each M_t invariant. Because $H^2(m)$ is assumed to contain non-constant functions, it is not difficult to see that for some $t>0$, M_t contains non-zero functions. Let f be such a function and note that not both $\chi_E f$ and $\chi_{X-E} f$ can be zero. Since both these functions are in M_t and since $M_t \subsetneqq H^2(m)$ for $t \geqq 0$, we see that $H^2(m)$ contains non-zero functions supported either on E or on $X-E$. Thus the proof of Theorem IV is complete.

A word of explanation concerning the hypothesis of Theorem IV is in order. If $\mathfrak{A}$ separates the points of X, then the hypothesis of Theorem IV follows from the hypothesis of Theorem III (see [7, p. 33]). However, examples show that $\mathfrak{A}$ need not always separate points; moreover, it is easy to see that on the occasions when this occurs there are representing measures m for $\mathfrak{A}$ such that m is not a point mass, m is not ergodic, and such that $H^2(m)$ contains only constant functions. We note that in numerous cases of particular interest $\mathfrak{A}$ separates points. For example, this happens when the flow is strictly ergodic and when it is minimal.

Observe that if m is a representing measure for $\mathfrak{A}$ on X such that $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(m)$, then $H^2(m)$ consists solely of constants if and only if m is a point mass. Observe also that if $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(m)$ then the space $H^2(m)$ has the property that no non-zero function in it vanishes in a set of positive measure if and only if each $H^p(m)$ has this property, $1 \leqq p \leqq \infty$ (see [12]). Thus we find that the following corollary is a consequence of Theorem IV and [12].

Corollary 3. 1. *Let m be a representing measure for $\mathfrak{A}$ on X which is not a point mass and such that $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(m)$. Then the following assertions are equivalent:*

(i) *m is ergodic;*

(ii) *for $1 \leqq p \leqq \infty$, no non-zero function in $H^p(m)$ can vanish on a set of positive measure;*

(iii) *$H^\infty(m)$ is a maximal weak-$*$ closed subalgebra of $L^\infty(m)$.*

Our final goal is to show that when the flow is strictly ergodic each representing measure for $\mathfrak{A}$ which is not a point mass on X is ergodic. To achieve this goal we prove the following result which is interesting in its own right.

Theorem V. *Suppose that the flow $(X, \mathbf{R})$ is strictly ergodic, and let μ be the unique invariant probability measure on X. Then C_0 is a maximal ideal in $\mathfrak{A}$ and μ is its (necessarily unique) representing measure.*

Proof. The proof is very similar to part (i) in the proof of Theorem I. Let $\mathfrak{A}_\mu=\{\varphi\in\mathfrak{A}|\langle\varphi, \mu\rangle=0\}$. Since μ is invariant the arguments in the proof of Theorem I show that $C_0\subseteqq\mathfrak{A}_\mu$. Therefore, to complete the proof, it clearly suffices to show $\mathfrak{A}_\mu\subseteqq C_0$. To this end, suppose φ is in $\mathfrak{A}_\mu$ but not in C_0 and choose a measure λ in $M(X)$ which annihilates C_0 while satisfying the equation $\langle\varphi, \lambda\rangle=1$. By Proposition 2 of [1], λ is analytic. Consequently, if $F(t)=\langle T_t\varphi, \lambda\rangle$ then as before F is the constant one. For positive τ we set $\varphi_\tau=\frac{1}{2r}\int_{-\tau}^{\tau}(T_\tau\varphi)dt$. Then for each such τ, $\langle\varphi_\tau, \lambda\rangle=\frac{1}{2\tau}\int_{-\tau}^{\tau}F(t)dt=1$ and so

$$\lim_{\tau\to\infty}\langle\varphi_\tau, \lambda\rangle=1.$$

In the proof of Theorem I we invoked the individual ergodic theorem to obtain a contradiction. However, that result has no bearing here. Instead we appeal to Théorème XV on page 107 of [11] and the strict ergodicity of the flow $(X, \mathbf{R})$ to conclude that the φ_τ converge pointwise (everywhere) to $\langle\varphi, \mu\rangle=0$. Since the φ_τ are uniformly bounded we see that

$$\lim_{\tau\to\infty}\langle\varphi_\tau, \lambda\rangle=0.$$

This contradiction completes the proof.

Theorem VI. *If the flow $(X, \mathbf{R})$ is strictly ergodic and if m is an arbitrary representing measure for $\mathfrak{A}$ on X which is not a point mass, then m is ergodic.*

Proof. Let μ be the unique invariant probability measure on X, recall that by Theorem V μ represents the maximal ideal C_0 in $\mathfrak{A}$, and consider the following two mutually exclusive and exhaustive cases.

Case 1. $\|\mu-m\|<2$. In this case, μ and m represent points in the same Gleason part of the maximal ideal space of $\mathfrak{A}$ and hence are mutually absolutely continuous [7, p. 143]. Thus since μ is ergodic, so is m.

Case 2. $\|\mu-m\|=2$. In this case, μ and m represent points in distinct Gleason parts of the maximal ideal space of $\mathfrak{A}$ and so μ and m are mutually singular [7, p. 144]. Since C_0 is the maximal ideal in $\mathfrak{A}$ represented by μ (Theorem V), the abstract Kolmogoroff—Krein Theorem [7, p. 135] implies that C_0 is dense in $H^2(m)$.

For each t in $\mathbf{R}$, let M_t be the closure of C_t in $L^2(m)$ and let P_t be the projection of $L^2(m)$ onto M_t. In the proof of Theorem IV it was shown that $\{P_t\}_{t\in\mathbf{R}}$ is a resolution of the identity whose Fourier—Stieltjes transform $\{V_t\}_{t\in\mathbf{R}}$ is a strongly continuous unitary representation of $\mathbf{R}$ on $L^2(m)$ which satisfies equation (3. 1). By equation (3. 1) we know that if E is any invariant Baire set, then χ_E leaves each M_t invariant. This is true in particular for M_0. However, the conclusion of the preceding paragraph is that $M_0=H^2(m)$. It follows from this and the fact that $H^2(m)$ contains no non-constant real-valued functions that if E is an invariant Baire set, then $m(E)$ is zero or one. Whence, in case 2 also, m is ergodic and the proof is complete.

§ **4.** One problem which arises at this point is to determine the structure of the maximal ideal space of $\mathfrak{A}$. We have been able to show that in the strictly ergodic case a point in the maximal ideal space of $\mathfrak{A}$, which is not in $X\cup\{C_0\}$, lies in a non-trivial Gleason part. Moreover, C_0 lies in a non-trivial Gleason part if and only if the unique invariant probability measure is supported on a non-trivial orbit. On the basis of these facts and what is known for spaces of generalized analytic functions [7, p. 171], we conjecture that in the strictly ergodic case, at least, the maximal ideal space of $\mathfrak{A}$ is homeomorphic to $X\times[0, 1]$ with the slice $X\times\{0\}$ identified to a point.[1)]

It appears that virtually all of the Helson—Lowdenslager invariant subspace theory [10] is valid in our setting and that an analysis similar to that in [13] may be developed to determine the Hilbert space representations of the algebra of analytic functions associated with a flow. We hope to pursue these matters in a future article.

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1) *Added in proof:* This conjecture is correct, at least if X is separable and the unique invariant measure is not concentrated on a periodic orbit. For the proof, see our paper: Function Algebras and Flows. II, which will appear in *Arkiv för Matematik.*

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