The Denjoy—Luzin theorem on trigonometric series

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Let $\{g_n\}$ be a sequence of transformations from the real line to the real line. Throughout this paper assume that f is periodic with period $p = b-a$ (>0), $f \in L^1[a, b]$, and set $f_n(x) = f(g_n(x))$. The sequence $\{g_n\}$ will be called an *A-sequence with respect to f* if the absolute convergence of $\Sigma c_n f_n(x)$ in a set of positive measure implies that $\Sigma |c_n|$ converges. The classical Denjoy—Luzin Theorem on trigonometric series states that $g_n(x) = nx + B_n$ is an A-sequence with respect to $f(x) = \cos x$, where $\{B_n\}$ is any sequence of numbers. The present author generalizes this result by considering the infimum of the averages of a function over sets of constant measure. The following result is a simple corollary. The sequence $g_n(x) = A_n x + B_n$ is an A-sequence for every periodic f that is essentially nonzero (i.e., nonzero almost everywhere) if and only if $\liminf |A_n| > 0$.

Theorem A (ORLICZ [1], p. 327). *The sequence $\{g_n\}$ is an A-sequence with respect to a periodic f if and only if*

$$\liminf_{n\to\infty} \int_E |f_n(x)|\, dx > 0$$

for every set E of positive measure.

Theorem B (FEJÉR [3], p. 48; [6], p. 49). *If $g_n(x) = A_n x + B_n$, where B_n are arbitrary real numbers and* $\lim |A_n| = \infty$, *then*

$$\lim_{n\to\infty} (\mu(E))^{-1} \int_E |f_n(x)|\, dx = (b-a)^{-1} \int_a^b |f(x)|\, dx$$

for every set E of positive measure.

Theorem 1. *The sequence $g_n(x) = A_n x + B_n$ is an A-sequence for every periodic f that is not essentially zero if and only if* $\lim |A_n| = \infty$.

Proof. *Sufficiency* (cf. [4], p. 84). Apply Theorems A and B.

Necessity. Let $f(x)=0$ on $(-1, 1)$, 1 on $(1, 3)$, and $f(x+4)=f(x)$. If $\liminf |A_n|<\infty$, then $\liminf |A_n\delta|<1$ for some $\delta>0$. Hence

$$\liminf_{n\to\infty} \int_0^\delta f(A_n x)\,dx=0.$$

Definition 1. If $h\in L^1[a, b]$, A is a measurable set, and $0<\delta\leqq\mu(A)$, then we define

$$m(h, \delta, A)=\inf_B \Big\{(\mu(B))^{-1}\int_B |h(t)|\,dt\Big\}$$

for measurable subsets B of A with $M(B)\geqq\delta$, and we define

$$M(h, \delta, A)=\text{supremum of the above set of averages.}$$

Also if $A=[a, b]$, we simply write $m(h, \delta)$ or $M(h, \delta)$ for any function h under consideration.

Remark 1. From Definition 1, we obtain

$$M(h, \delta)\uparrow \|h\|_\infty\equiv \operatorname{ess\,sup}|h| \quad \text{as} \quad \delta\downarrow 0,$$
$$m(h, \delta)\downarrow \|h\|_0\equiv \operatorname{ess\,inf}|h| \quad \text{as} \quad \delta\downarrow 0.$$

Lemma 1. *Let $h\in L^1[a, b]$. Then h is essentially nonzero if and only if for every measurable set A $m(f, \delta, A)>0$ whenever $0<\delta\leqq\mu(A)<\infty$.*

Proof. Assume h is essentially nonzero and $0<\delta\leqq\mu(A)<\infty$. Then for $B\subset A$ and $\mu(B)\geqq\delta$ we set

$$H_n=\{x\in B: |h(x)|>n^{-1}\}$$

and choose N such that $\mu(H_N)>\mu(B)/2$. Then, we obtain

$$\int_B |h(t)|\,dt\geqq\mu(H_N)/N\geqq\mu(B)/2N.$$

Lemma 2 ([5], p. 315; [2], p. 1245). *If g is a strictly monotonic absolutely continuous function on $[a, b]$ with range $[c, d]$ $(\{g(x): x\in[a, b]\}=[c, d])$, then for every measurable set $E\subset[a, b]$*

a)
$$\int_E f(g(x))|g'(x)|\,dx=\int_{g(E)} f(y)\,dy$$

and

b)
$$\delta m(g', \delta)\leqq\int_E |g'(x)|\,dx=\mu(gE)\leqq\delta M(g', \delta) \quad \text{if} \quad \mu(E)=\delta.$$

From Definition 1 we also obtain

Lemma 3. *If n is a positive integer and $0<\delta\leqq np$, then*

$$m(f, \delta, [0, np])=m(f, \delta/n).$$

Definition 2. If e is a real number we let e^* and e_* denote respectively the least integer greater than or equal to e and the greatest integer less than or equal to e.

Theorem 2. *Let g be a strictly monotonic absolutely continuous function with domain $[a, b]$ and range $[c, d]$. If $e = \|g'\|_1/p = (d-c)/p$ and $f\circ g$ is the function $f(g(x))$, then*

$$\frac{m(g',\delta)}{\|g'\|_\infty} m\left(f, \frac{\delta m(g',\delta)}{e^*}\right) \leqq m(f\circ g, \delta) \leqq \frac{M(g',\delta)}{\|g'\|_0} m\left(f, \frac{\delta M(g',\delta)}{e_*}\right)$$

where for the left inequality we assume $\|g'\|_\infty$ is finite and $0<\delta\leqq p$, and for the right inequality we assume $\|g'\|_0$ and e_* are nonzero and $0<\delta M(g',\delta)/e_*\leqq p$.

Proof. By Lemmas 2 and 3

$$m(f\circ g, \delta)\|g'\|_\infty \geqq m((f\circ g)g', \delta) \geqq m(g',\delta)m(f, \delta m(g',\delta), [c, d]) \geqq$$

$$\geqq m(g',\delta)m(f, \delta m(g',\delta), [c, c+e^*p]) = m(g',\delta)m(f, \delta m(g',\delta)/e^*).$$

The proof of the right half is similar.

Corollary 1. *If $\{g_n\}$ is a sequence of strictly monotonic absolutely continuous functions with domain $[a, b]$, and also for each g_n and each δ such that $0<\delta\leqq p$, we have*

$$(\|g_n'\|_\infty)^* \leqq K(\delta)m(g_n', \delta)$$

where $K(\delta)$ is a positive constant dependent only on δ, then $\{g_n\}$ is an A-sequence for every periodic f that is essentially nonzero.

Corollary 2. *If $A\neq 0$, $g(x) = Ax+B$, and we set $e=|A|$, then*

$$m(f, \delta e/e^*) \leqq m(f\circ g, \delta) \leqq m(f, \delta e/e_*) \tag{1}$$

where for the left inequality we assume $0<\delta\leqq p$, and for the right inequality we assume $e\geqq 1$ and $0<\delta e/e_*\leqq p$.

Corollary 3. *The sequence $g_n(x) = A_n x+B_n$ is an A-sequence for every periodic f that is essentially nonzero if and only if* $\liminf |A_n|>0$.

Proof. *Sufficiency.* If $\liminf A_n>0$, then $\liminf A_n/A_n^*>0$. Now, our result follows from Theorem A, Lemma 1, and Corollary 2.

Necessity. If $f(x)=x$ on $[0, 1]$, then

$$\liminf_{n\to\infty} \int_0^\delta |f(A_n x)|\, dx = (\delta^2/2) \liminf_{n\to\infty} |A_n|$$

and so our result follows by Theorem A.

Remark 2. The use of e^* and e_* is necessary in Theorem 2 and its corollaries. In fact, Corollary 2 is best possible in a certain sense. For example, if $f(x)=x$ on $[0, 1]=[a, b]$ and $g(x)=ex$ for a noninteger $e>0$, then equality is attained in the left side of (1) for $\delta < e^*(1-e_*/e)$. Similarly, if $f(x) = 1-x$ on $[0, 1]=[a, b]$ and $g(x)=ex$ for a noninteger $e>1$, then equality is attained in the right side of (1) for $\delta < e_*(e^*/e-1)$.

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