The centroid of a semigroup

By MARIO PETRICH in University Park (Pennsylvania, U.S.A.)

1. Introduction and summary. The centroid of a ring R is defined as the centralizer in the ring of all endomorphisms of the additive group of R of the ring generated by all left and right multiplications [10]. It reduces essentially to the center of R if R has an identity element. We borrow the multiplicative part of the definition of a centroid and apply this notion to the theory of semigroups. Hence the centroid $Z(S)$ of a semigroup S is the semigroup under composition of all transformations on S which are simultaneously left and right translations of S (written, say, as left operators). We exploit this concept for two principal purposes. The first one is connected with a generalization of inner automorphisms and the second one is a consideration of the congruence on the semigroup whose classes are the orbits of the group of units of the centroid. We study this congruence in detail for several classes of semigroups, but the most fruitful classes turn out to be cancellative and commutative cancellative semigroups. Based on this congruence, certain semigroups are isomorphic to a Schreier extension of an abelian group and another semigroup and can be embedded into a wreath product of these.

Section 2 is a preliminary one and contains most of the background needed throughout the paper. In Section 3 we introduce a generalization of inner automorphisms, and for a wide class of semigroups prove a theorem which in the case of groups reduces to the familiar relationship between inner automorphisms and the center. In Section 4 we study the congruence σ_S on a semigroup S whose classes are the orbits of the group of units of the centroid $GZ(S)$ of S. We prove that a semigroups in which $GZ(S)$ acts simply transitively on each σ_S-class is isomorphic to a Schreier extension of $GZ(S)$ by S/σ_S. We also establish an isomorphism theorem for this representation of S. For such a Schreier extension, we prove in Section 5 that for a wide class of semigroups, the extension can be embedded (or densely embedded) in the wreath product of some related semigroups. For cancellative semigroups S, in Section 6 we find an expression for σ_S in terms of elements of S, and for every element of S, find a copy of $Z(S)$ defined on a subset of S with a new multiplication. We also consider an example exhibiting interesting features in this context. Finally, in Section 7 we deal with commutative cancellative semigroups.

Every such semigroup is isomorphic to a Schreier extension of an abelian group and a commutative cancellative semigroup Q in which $aQ=bQ$ always implies $a=b$, and conversely. We further extablish several properties of this decomposition of S and of the congruence σ_S. For the case of the additive semigroups of all nonnegative or of all positive integers, we compute all functions figuring in the Schreier extension. It should be noted already that what we call a Schreier extension is close but not identical with the concept of the Schreier product used in [12]. Furthermore, since we often write functions on the left, we apply the "left version" of the wreath product and use the notation "*wl*" instead of "*wr*".

2. Preliminaries. We begin by recalling the concepts needed throughout the paper; for undefined terms and notation the reader is referred to [2]. Let S be a semigroup and let x, y be arbitrary elements of S. A function λ on S written on the left is a *left translation* of S if $\lambda(xy)=(\lambda x)y$; a function ϱ on S written on the right is a *right translation* if $(xy)\varrho=x(y\varrho)$; λ and ϱ are *linked* and we say that (λ, ϱ) is a *bitranslation* of S if $x(\lambda y)=(x\varrho)y$; λ and ϱ are *permutable* if $(\lambda x)\varrho=\lambda(x\varrho)$. The set $\Lambda(S)$ of all left translations of S under the composition $(\lambda\lambda')x=\lambda(\lambda' x)$ is a semigroup; the set $P(S)$ of all right translations is a semigroup under the composition $x(\varrho\varrho')=(x\varrho)\varrho'$. The set $\Omega(S)$ of all bitranslations of S with the multiplication induced by the direct product $\Lambda(S)\times P(S)$ is a semigroup called the *translational hull* of S. For $a\in S$, the functions λ_a and ϱ_a defined by $\lambda_a x=ax$ $x\varrho_a=xa$, are, respectively, the *inner left translation* and the *inner right translation* of S induced by a; $\pi_a=(\lambda_a, \varrho_a)$ is the *inner bitranslation* of S induced by a; $\Pi(S)=\{\pi_a|a\in S\}$ is the *inner part* of $\Omega(S)$.

It is easy to verify that

$$(1)\qquad (\lambda, \varrho)\pi_a=\pi_{\lambda a}, \quad \pi_a(\lambda, \varrho)=\pi_{a\varrho} \qquad (a\in S, (\lambda, \varrho)\in\Omega(S)),$$

which implies that $\Pi(S)$ is an ideal of $\Omega(S)$; one verifies similarly that $\Gamma(S)=\{\lambda_a|a\in S\}$ is a left ideal of $\Lambda(S)$ and that $\Delta(S)=\{\varrho_a|a\in S\}$ is a right ideal of $P(S)$. The projections

$$\pi_\Lambda:(\lambda, \varrho)\to\lambda, \quad \pi_P:(\lambda, \varrho)\to\varrho \qquad ((\lambda, \varrho)\in\Omega(S))$$

are homomorphisms, let

$$\tilde{\Lambda}(S)=\pi_\Lambda\Omega(S), \quad \tilde{P}(S)=\pi_P\Omega(S).$$

By $C(S)$ we denote the *center* of S, and if S has an identity, $G(S)$ denotes the *group of units* (invertible elements of S). As a generalization of the center of a semigroup, we borrow the following concept from the theory of rings: the *centroid* of S, denoted by $Z(S)$, is the set of all functions ζ on S satisfying $\zeta(xy)=(\zeta x)y=x(\zeta y)$ $(x, y\in S)$. It follows immediately that $Z(S)$ is a subsemigroup of $\Lambda(S)$

and is the centralizer of the set of all inner left and inner right translations of S if both of these are written as left operators (definition in [10], V, § 4).

If several operators are applied to S, we will retain the parentheses only around S, e.g., we write $C\Omega(S)$ instead of $C(\Omega(S))$, etc. If S has an identity e, then for any $\lambda \in \Lambda(S)$ and $\varrho \in P(S)$, we have $\lambda = \lambda_{\lambda e}$ and $\varrho = \varrho_{e\varrho}$, so $\Gamma(S) = \Lambda(S)$, $\Delta(S) = P(S)$, $\Omega(S) = \Pi(S)$, and $Z(S)$ can be identified with $C(S)$.

The transformation ι_S written on the left is the identity both of $Z(S)$ and $\Lambda(S)$; the pair (ι_S, ι_S), where the second ι_S is written on the right, is the identity of $\Omega(S)$. The groups $GZ(S)$ and $G\Omega(S)$ will play a central role in our investigations.

3. Generalized inner automorphisms. We will now introduce a class of automorphisms of an arbitrary semigroup S which in the case of groups reduces exactly to the set of all inner automorphisms. Another generalization of an inner automorphism was introduced by DUBREIL [4] and was intensively studied by CROISOT [3] for cancellative semigroups and by THIERREN [14] for reductive semigroups. We will see in Section 6 by an example that these two generalizations of inner automorphisms of a group are very different.

3. 1 Proposition. *Let S be any semigroup, $(\lambda, \varrho) \in G\Omega(S)$, and assume that λ and ϱ are permutable. Then $\lambda^{-1} \in \Lambda(S)$, λ^{-1} is permutable with ϱ, and the function $\delta_{(\lambda,\varrho)}$ defined by:*

$$s\delta_{(\lambda,\varrho)} = (\lambda^{-1}s)\varrho \quad (s \in S) \tag{2}$$

is an automorphism of S.

Proof. Since both λ and ϱ are permutations on the set S, so is $\delta_{(\lambda,\varrho)}$. For any $x, y \in S$, we obtain $\lambda[(\lambda^{-1}x)y] = (\lambda\lambda^{-1}x)y = xy = (\lambda\lambda^{-1})(xy) = \lambda[\lambda^{-1}(xy)]$ which implies $(\lambda^{-1}x)y = \lambda^{-1}(xy)$. Using this, we compute

$$\begin{aligned}(xy)\delta_{(\lambda,\varrho)} &= [\lambda^{-1}(xy)]\varrho = [(\lambda^{-1}x)y]\varrho = (\lambda^{-1}x)(y\varrho) = (\lambda^{-1}x)[(\lambda\lambda^{-1}y)\varrho] = \\ &= (\lambda^{-1}x)\{\lambda[(\lambda^{-1}y)\varrho]\} = [(\lambda^{-1}x)\varrho][(\lambda^{-1}x)\varrho] = (x\delta_{(\lambda,\varrho)})(y\delta_{(\lambda,\varrho)})\end{aligned}$$

as required. Further, $\lambda[(\lambda^{-1}x)\varrho] = (\lambda\lambda^{-1}x)\varrho = x\varrho = \lambda[\lambda^{-1}(x\varrho)]$ and thus $(\lambda^{-1}x)\varrho = \lambda^{-1}(x\varrho)$.

In view of 3. 1, we may write $x\delta_{(\lambda,\varrho)} = \lambda^{-1}x\varrho$ without ambiguity. If S has an identity element, we may define the *inner automorphism* ε_a induced by an element $a \in G(S)$ by the usual formula:

$$s\varepsilon_a = a^{-1}sa \quad (s \in S). \tag{3}$$

In such a case, $\delta_{(\lambda,\varrho)} = \varepsilon_{e\varrho}$ where e is the identity of S. Conversely, for $a \in G(S)$, we have $\delta_{\pi_a} = \varepsilon_a$. It is then natural to introduce the following notion.

3. 2 Definition. With the notation of 3. 1, $\delta_{(\lambda,\varrho)}$ is the *generalized inner automorphism* of S induced by (λ, ϱ).

The *group of all automorphisms* of a semigroup S will be denoted by $\mathscr{A}(S)$, the set of *all generalized inner automorphisms* by $\mathscr{I}(S)$. As we have seen above, in the case that S has an identity, $\mathscr{I}(S)$ coincides with the group of inner automorphisms of S. The introduced terminology is further justified by a theorem valid for a large class of semigroups, which reduces to the familiar connection between $\mathscr{I}(S)$ and $C(S)$ when S is a group. For this we need some preliminaries. Recall that S is *weakly reductive* if the mapping $\pi: a \to \pi_a$ $(a \in S)$ is one-to-one (and thus an isomorphism of S onto $\Pi(S)$); S is *globally idempotent* if $S^2 = S$.

3. 3 Lemma. *The following statements concerning a semigroup S which is either weakly reductive or globally idempotent are true:*

a) *If $(\lambda, \varrho), (\lambda', \varrho') \in \Omega(S)$, then λ and ϱ' are permutable.*
b) $C\Omega(S) = \{(\lambda, \varrho) \in \Lambda(S) \times P(S) \mid \lambda s = s\varrho$ *for all* $s \in S\}$.
c) $\pi_{\Lambda|_{C\Omega(S)}}$ *is an isomorphism of $C\Omega(S)$ onto $Z(S)$.*

We omit the proof. In a different form, a) is mentioned in CLIFFORD [1].

Under the hypotheses of the lemma, the centroid of S can be identified with the center of $\Omega(S)$, and this case is the most interesting one. In particular $Z(S)$ is then commutative. This represents a slight improvement over ([10], V, § 4, Proposition 1) where $\mathfrak{Z}_r(\mathfrak{A}) = 0$ or $\mathfrak{Z}_l(\mathfrak{A}) = 0$ can be replaced by vanishing of the double annihilator $\{a \in \mathfrak{A} \mid ax = xa = 0$ for all $x \in \mathfrak{A}\}$. Furthermore, part a) shows that $\delta_{(\lambda,\varrho)}$ is defined for every element $(\lambda, \varrho) \in G\Omega(S)$.

The proof of the following lemma is a straightforward verification and is omitted.

3. 4. Lemma. *Let θ be an isomorphism of a semigroup S onto a semigroup T. Then the function $\bar{\theta}$ defined by:*

$$\bar{\theta}: (\lambda, \varrho) \to (\bar{\lambda}, \bar{\varrho}) \qquad ((\lambda, \varrho) \in \Omega(S)) \tag{4}$$

where

$$\bar{\lambda} t = [\lambda(t\theta^{-1})]\theta, \quad t\bar{\varrho} = [(t\theta^{-1})\varrho]\theta \qquad (t \in T) \tag{5}$$

is an isomorphism of $\Omega(S)$ onto $\Omega(T)$.

The following is the principal result of this section. Recall the notation (2), (3), (4), (5), $\mathscr{A}(S)$, $\mathscr{I}(S)$.

3. 5 Theorem. *Let S be a semigroup which is either weakly reductive or globally idempotent. Then the mapping*

$$\chi: (\lambda, \varrho) \to \delta_{(\lambda,\varrho)} \qquad ((\lambda, \varrho) \in G\Omega(S))$$

is a homomorphism of $G\Omega(S)$ onto $\mathcal{I}(S)$ with kernel $GC\Omega(S)$ so that

$$G\Omega(S)/GC\Omega(S)\cong\mathcal{I}(S).$$

Moreover, $\pi_\Lambda GC\Omega(S)=GZ(S)$ and $\bar{\delta}_{(\lambda,\varrho)}=\varepsilon_{(\lambda,\varrho)}$ for all $(\lambda,\varrho)\in G\Omega(S)$.

Proof. Using part a) of 3. 3, for any $(\lambda,\varrho),(\varphi,\psi)\in G\Omega(S)$ and $s\in S$, we have

$$s\delta_{(\lambda,\varrho)}\delta_{(\varphi,\psi)}=\varphi^{-1}(\lambda^{-1}s\varrho)\psi=(\lambda\varphi)^{-1}s(\varrho\psi)=s\delta_{(\lambda\varphi,\varrho\psi)}=s\delta_{(\lambda,\varrho)(\varphi,\psi)}$$

and hence χ is a homomorphism. Let $(\lambda,\varrho)\in G\Omega(S)$. Then $(\lambda,\varrho)\in\ker\chi$ if and only if $\delta_{(\lambda,\varrho)}=(\iota_{\Omega(S)},\iota_{\Omega(S)})$, equivalently $\lambda^{-1}s\varrho=s$ for all $s\in S$, which can be written as $s\varrho=\lambda s$ for all $s\in S$. By part b) of 3. 3, the latter is equivalent to $(\lambda,\varrho)\in C\Omega(S)$. Consequently $(\lambda,\varrho)\in\ker\chi$ if and only if $(\lambda,\varrho)\in G\Omega(S)\cap C\Omega(S)=GC\Omega(S)$ as required. The equality $\pi_\Lambda GC\Omega(S)=CZ(S)$ follows from part c) of 3. 3. For $(\lambda,\varrho)\in G\Omega(S)$ and $(\varphi,\psi)\in\Omega(S)$, we have

$$(\varphi,\psi)\bar{\delta}_{(\lambda,\varrho)}=(\bar{\varphi},\bar{\psi})$$

where for any $s\in S$,

$$\bar{\varphi}s=[\varphi(s\delta^{-1}_{(\lambda,\varrho)})]\delta_{(\lambda,\varrho)}=\lambda^{-1}[\varphi(\lambda s\varrho^{-1})]\varrho=(\lambda^{-1}\varphi\lambda)s$$

and analogously $s\bar{\psi}=s(\varrho^{-1}\psi\varrho)$, which implies

$$(\bar{\varphi},\bar{\psi})=(\lambda^{-1}\varphi\lambda,\varrho^{-1}\psi\varrho)=(\lambda,\varrho)^{-1}(\varphi,\psi)(\lambda,\varrho)=(\varphi,\psi)\varepsilon_{(\lambda,\varrho)}.$$

Consequently $\bar{\delta}_{(\lambda,\varrho)}=\varepsilon_{(\lambda,\varrho)}$ proving the last assertion of the theorem.

It is clear that for the case when S is a group, the foregoing result reduces to the familiar theorem in group theory. If S is weakly reductive, we may identify $\Pi(S)$ and S; the last assertion of the above theorem then states that the generalized inner automorphisms are the restrictions of inner automorphisms of $\Omega(S)$ to S. In particular, for the case of $S=\mathfrak{F}(\mathfrak{M},\mathfrak{N})$ and $\Omega(S)=\mathfrak{L}(\mathfrak{M},\mathfrak{N})$, with the obvious identifications, where $\mathfrak{M}$, $\mathfrak{N}$ is a pair of dual vector spaces over a (not necessarily commutative) field Φ, the set of all "quasi-inner automorphisms" of MALCEV forms a proper subgroup of $\mathcal{I}(S)$ as shown by ROSENBERG ([13], p. 125).

It should be noted that in general $GC\Omega(S)\neq CG\Omega(S)$. For example, if $S=\mathcal{M}^\circ(G;I,I;\Delta)$ is a Brandt semigroup, it can be shown that the equality fails to occur only in the case when G is trivial and I has exactly 2 elements (stated by HOEHNKE [9], Satz 1), in which case $G\Omega(S)$ is of order 2 and $GC\Omega(S)$ is trivial. In section 6 we will discuss a similar example when S is cancellative. Further properties of the concepts discussed above, as well as the proofs omitted here, can be found in [11].

4. The congruence σ_S. We have seen in the preceding section that for a semigroup S with identity, $Z(S)$ can be identified with $C(S)$. If S is also a group, $GZ(S)=Z(S)$ and we may consider S as a Schreier extension of $Z(S)$ by $S/Z(S)$. The following will show how this can be generalized to the situation in which S is a semigroup satisfying relatively weak conditions.

4.1 Definition. For any semigroup S, let σ_S be the equivalence relation on S whose classes are the orbits of $GZ(S)$. Hence for any $a, b \in S$, $a\sigma_S b$ if and only if there exists $\lambda \in GZ(S)$ such that $a=\lambda b$.

It is immediate that σ_S is a congruence relation on S. We will often write σ instead of σ_S if there is no danger of confusion. If S has an identity element, then the classes of σ_S coincide with the orbits of $GC(S)$. In particular, for any semigroup S and $a, b \in S$, $\pi_a \sigma_{\Omega(S)} \pi_b$ if and only if $\pi_a=(\lambda, \varrho)\pi_b$ for some $(\lambda, \varrho) \in GC\Omega(S)$, which is in turn equivalent to $\pi_a=\pi_{\lambda b}$ in view of (1). Consequently, for a weakly reductive semigroup S, we have $\pi_a \sigma_{\Omega(S)} \pi_b$ if and only if $a\sigma_S b$. In such a case, if we identify $\Pi(S)$ with S, we may write $\sigma_{\Omega(S)}|_{\Pi(S)}=\sigma_S$.

It follows from the definition of σ that $GZ(S)$ acts as a transitive group of permutations on each σ-class. If $GZ(S)$ acts simply transitively on each σ-class, we will show that S can be expressed as a Schreier extension of $GZ(S)$ by S/σ in the sense of the following (cf. [12])

4.2 Definition. Let Q be a semigroup, Φ be an (additively written) abelian group, and let $[\ ,\ ]: Q\times Q \to \Phi$ be a function satisfying

$$[a, b]+[ab, c] = [a, bc]+[b, c] \qquad (a, b, c \in Q). \tag{6}$$

Let $S = Q\times\Phi$ together with the multiplication

$$(a, \alpha)(b, \beta) = (ab, [a, b]+\alpha+\beta).$$

(It is easy to verify that (6) is equivalent to associativity.) We call S a *Schreier extension of Φ by Q* and denote it by $(Q, \Phi, [\ ,\ ])$.

We are now in a position to prove the desired result. It should now be noted that a sufficient condition on a semigroup S in order that $GZ(S)$ act simply transitively on each σ-class is weak cancellation, viz., the conjunction of $xa=ya$ and $ax=ay$ implies $x=y$. For if $\lambda \in GZ(S)$ and $a=\lambda b=\lambda' b$, then for any $x \in S$, $(\lambda x)a=x(\lambda a)= =x(\lambda' a)=(\lambda' x)a$ and analogously $a(\lambda x)=a(\lambda' x)$, which by weak cancellation yields $\lambda=\lambda'$.

4.3 Theorem. *A semigroup S for which $GZ(S)$ acts simply transitively on each σ-class is isomorphic to a Schreier extension of $GZ(S)$ by S/σ.*

Proof. Let $Q=S/\sigma$ and arbitrarily choose a system $\{z_a\}_{a\in Q}$ of representatives of σ-classes. Letting $\Phi=GZ(S)$, define a function $[\ ,\ ]: Q\times Q \to \Phi$ by the requirement: $[a, b]\in\Phi$ for which $[a, b]z_c=z_a z_b$ where $z_c \sigma z_a z_b$. Next define a function χ on $Q\times\Phi$ by:

$$\chi: x \to (a, \lambda) \quad \text{if} \quad x \in \sigma\text{-class } a \text{ and } \lambda z_a = x.$$

The hypothesis on Φ implies that χ is a bijection of S onto $Q\times\Phi$. For $x\chi=(a, \lambda)$, $y\chi=(b, \mu)$, $(xy)\chi=(c, \nu)$, we obtain

$$(\lambda\mu[a, b])z_c=\lambda\mu\{[a, b]z_c\}=\lambda\mu(z_a z_b)=(\lambda z_a)(\mu z_b)=xy$$

which by simple transitivity implies $\lambda\mu[a, b]=\nu$. Writing Φ additively, we obtain

$$(x\chi)(y\chi) = (a, \lambda)(b, \mu) = (ab, [a, b]+\lambda+\mu) = (xy)\chi$$

which shows that χ is a homomorphism. Condition (6) on $[\ ,\ ]$ is equivalent to associativity and hence follows here from the associativity in S. Therefore χ is an isomorphism of S onto $(Q, \Phi, [\ ,\ ])$ as required.

If S has an identity element 1, then 1 can be chosen as the representative of its σ-class e, which then yields $[a, e]=[e, a]=0$ where 0 in additive notation stands for the identity function. This is the usual "initial condition" imposed on "Schreier extensions" in group, ring, or semigroup theory (see RÉDEI [12]). Conversely, if $S=(Q, \Phi, [\ ,\ ])$, e is the identity of Q, and $[a, e]=[e, a]=0$ for all $a\in Q$, then $(e, 0)$ is the identity of S. In such a case, the mapping $\alpha\to(e, \alpha)$ $(\alpha\in\Phi)$ embeds Φ into S in a natural way. For the Schreier extensions of semigroups S and S' constructed above, we have the following isomorphism criterion.

4. 4 Theorem. *Let S and S' satisfy the condition in* 4. 3, *and let $T=(Q, \Phi, [\ ,\ ])$, $T'=(Q', \Phi', [\ ,\ ]')$ be the isomorphic copies of S and S', respectively, as in* 4. 3. *Then $S\cong S'$ if and only if there exists an isomorphism θ of Q onto Q' and for each $a\in Q$ there is a bijection η_a of Φ onto Φ' such that*

$$\eta_a\alpha+\eta_b\beta+[a\theta, b\theta]' = \eta_{ab}([a, b]+\alpha+\beta) \qquad (a, b\in Q,\quad \alpha, \beta\in\Phi). \tag{7}$$

Proof. Suppose first that $S\cong S'$. Then there exists an isomorphism ψ of T onto T'. It is easy to see that $(a, \alpha)\sigma_T(b, \beta)$ if and only if $a=b$. It follows from 3. 4 that an isomorphism preserves σ-classes. Consequently there exists an isomorphism θ of Q onto Q' making the diagram

$$\begin{tikzcd} T \arrow[r, "\psi"] \arrow[d, "\varphi"'] & T' \arrow[d, "\varphi'"] \\ T/\sigma_T \arrow[d, "\xi"'] & T'/\sigma_{T'} \arrow[d, "\xi'"] \\ Q \arrow[r, "\theta"] & Q' \end{tikzcd}$$

commutative, where φ and φ' are canonical homomorphisms, and $\langle a, \alpha\rangle\xi=a$, $\langle a', \alpha'\rangle\xi'=a'$ for σ-classes $\langle a, \alpha\rangle$ and $\langle a', \alpha'\rangle$ of T and T', respectively. Hence for any $(a, \alpha)\in T$ and $(a, \alpha)\psi=(b, \beta)$, we obtain $(a, \alpha)\varphi\xi\theta=\langle a, \alpha\rangle\xi\theta=a\theta$ and $(a, \alpha)\psi\varphi'\xi'=$

$=(b, \beta)\varphi'\xi' = \langle b, \beta\rangle\xi' = b$, and thus $a\theta = b$. Now writing β in the form $\eta_a\alpha$, we see that

$$(8) \qquad (a, \alpha)\psi = (a\theta, \eta_a\alpha) \qquad (a\in Q, \quad \alpha\in\Phi)$$

where η_a is a mapping of Φ into Φ' for every $a\in Q$. Thus for any (a, α), $(b, \beta)\in T$, we have

$$(a, \alpha)\psi(b, \beta)\psi = (a\theta, \eta_a\alpha)(b\theta, \eta_b\beta) = \big((a\theta)(b\theta), [a\theta, b\theta]' + \eta_a\alpha + \eta_b\beta\big),$$

$$[(a, \alpha)(b, \beta)]\psi = (ab, [a, b] + \alpha + \beta)\psi = \big((ab)\theta, \eta_{ab}([a, b] + \alpha + \beta)\big)$$

which implies (7). For any $a\in Q$ and $\alpha'\in\Phi'$ there exists $\alpha\in\Phi$ such that $(a, \alpha)\psi = (a\theta, \alpha')$ since both ψ and θ are onto. By (8), we must have $\eta_a\alpha = \alpha'$ proving that η maps Φ onto Φ'. If $\eta_a\alpha = \eta_a\beta$, then by (8), we have $(a, \alpha)\psi = (a, \beta)\psi$ so that $\alpha = \beta$ since ψ is one-to-one. Thus η_a is a bijection.

Conversely, if the functions θ and η_a are given as in the theorem, then ψ given by (8) is easily seen to be an isomorphism of T onto T', which in turn implies the existence of an isomorphism of S onto S'.

We have seen in the proof that the converse is valid without $Q = S/\sigma_S$, $Q' = S'/\sigma_{S'}$. Hence for arbitrary Schreier extensions $T = (Q, \Phi, [\ ,\ ])$ and $T' = (Q', \Phi', [\ ,\ ]')$, 4. 4 furnishes sufficient conditions for their isomorphism. However, they are in general not necessary, for it could happen that $T\cong T'$ but $Q\not\cong Q'$. Furthermore, in the case of 4. 4, Φ and Φ' are isomorphic which is not explicit in these conditions.

5. A dense embedding. We will now establish that for a left or right reductive globally idempotent semigroup Q, a Schreier extension of an abelian group Φ by Q can be embedded (and in some special way) into several semigroups which are wreath products of a semigroup of transformations on Q and Φ. First we recall the pertinent definitions.

If B is an ideal of a semigroup A, then A is an *ideal extension* of B; if in addition the equality congruence on A is the only congruence on A whose restriction to B is the equality congruence on B, then A is a *dense extension* of B; if also A is, under inclusion, a maximal dense extension of A, then B is a *densely embedded ideal* of A. A subsemigroup C of A is a *densely embedded subsemigroup* if C is a densely embedded ideal of its idealizer $i_A(C)$ in A. An isomorphism φ of a semigroup D into A is a *dense embedding* if $D\varphi$ is a densely embedded subsemigroup of A, and we say that D can be *densely embedded* in A. Using the concepts of a left ideal, left idealizer, etc., one defines analogously an *l-densely embedded* ideal, subsemigroup, embedding etc. For an extensive discussion concerning these concepts, see GLUSKIN [5], [6].

Let X be a nonempty set, P a semigroup of transformations on X, written on the left, and let G be a group. On $S = P \times G$ define a multiplication by

$$(\alpha, \varphi)(\alpha', \varphi') = (\alpha\alpha', \varphi^{\alpha} \cdot \varphi')$$

where $(\varphi^{\alpha} \cdot \varphi')x = (\varphi\alpha x)(\varphi' x)$ $(x \in X)$. Then S is a semigroup called the *(left) wreath product* of P and G and will be denoted by $PwlG$.

The following discussion is motivated by ([6], Section 1).

Let Q be a right reductive semigroup, i.e., $ax=bx$ for all $x \in Q$ implies $a=b$, let Φ be an abelian group, and let $S=(Q, \Phi, [\ ,\ ])$ be a Schreier extension of Φ by Q.

Let $\lambda \in \Lambda(S)$; then for any $(a, \alpha) \in S$ we have

$$\lambda(a, \alpha) = (\xi(a, \alpha), \theta(a, \alpha)) \tag{9}$$

for some functions ξ and θ. Hence

$$[\lambda(a, \alpha)](b, \beta) = (\xi(a, \alpha), \theta(a, \alpha))(b, \beta) = (\xi(a, \alpha)b, [\xi(a, \alpha), b] + \theta(a, \alpha) + \beta),$$

$$\lambda[(a, \alpha)(b, \beta)] = \lambda(ab, [a, b] + \alpha + \beta) = (\xi(ab, [a, b] + \alpha + \beta), \theta(ab, [a, b] + \alpha + \beta)),$$

and thus

$$\xi(a, \alpha)b = \xi(ab, [a, b] + \alpha + \beta), \tag{10}$$

$$[\xi(a, \alpha), b] + \theta(a, \alpha) + \beta = \theta(ab, [a, b] + \alpha + \beta). \tag{11}$$

We substitute β in (10) by $\beta - [a, b] - \alpha$ and obtain $\xi(a, \alpha)b = \xi(ab, \beta)$. Since this is true for all $\alpha, \beta \in \Phi$, we also have $\xi(a, \beta)b = \xi(ab, \beta)$, and thus $\xi(a, \alpha)b = \xi(a, \beta)b$ for all $b \in Q$. By right reductivity of Q, we conclude that $\xi(a, \alpha) = \xi(a, \beta)$, i.e., $\xi(a, \alpha)$ is independent of α and we may write ξa instead of $\xi(a, \alpha)$. But then (10) yields $(\xi a)b = \xi(ab)$ so that $\xi \in \Lambda(Q)$.

Now (11) implies

$$\theta(a, \alpha) = \theta(ab, [a, b] + \alpha + \beta) - [\xi a, b] - \beta \tag{12}$$

which for $\beta = -[a, b] - \alpha$ becomes

$$\theta(a, \alpha) = \theta(ab, 0) - [\xi a, b] + [a, b] + \alpha. \tag{13}$$

For $\alpha = 0$, (11) takes on the form

$$\theta(a, 0) = \theta(ab, 0) - [\xi a, b] + [a, b]. \tag{14}$$

Now let $\eta a = \theta(a, 0)$ for all $a \in Q$. Substituting $\theta(ab, 0)$ from (14) into (13) in the new notation we have

$$\theta(a, \alpha) = \eta a + \alpha \qquad ((a, \alpha) \in S). \tag{15}$$

Further, (15) substituted into (12) yields $\eta a+\alpha = \eta(ab)+[a, b]+\alpha+\beta-[\xi a, b]-\beta$ and thus η satisfies the condition

$$\eta a-\eta(ab) = [a, b]-[\xi a, b] \qquad (a, b\in Q). \tag{16}$$

Hence (9) becomes

$$\lambda(a, \alpha) = (\xi a, \eta a+\alpha) \qquad ((a, \alpha)\in S). \tag{17}$$

5.1 Theorem. *With the notation introduced, define a function* ψ *by:*

$$\psi:\lambda\to(\xi, \eta) \qquad (\lambda\in \Lambda(S)) \tag{18}$$

where $\xi\in\Lambda(Q)$ *and* $\eta\in G^Q$ *satisfy* (16)*, and* λ *satisfies* (17). *Then* ψ *embeds* $\Lambda(S)$ *into* $\Lambda(Q)wl\Phi$. *In addition, if* Q *is globally idempotent, then* $\Lambda(S)\psi$ *is the largest subsemigroup of* $\Lambda(Q)wl\Phi$ *containing* $\Gamma(S)\psi$ *as a left ideal.*

Proof. If $\lambda\psi=(\xi, \eta)$ and $\lambda'\psi=(\xi', \eta')$, then for any $(a, \alpha)\in S$, $\lambda\lambda'(a, \alpha)= = \lambda(\xi' a, \eta' a+\alpha) = (\xi\xi' a, \eta\xi' a+\eta' a+\alpha)$ which in the multiplicative notation can be written as $\lambda\lambda'(a, \alpha) = (\xi\xi' a, (\eta^{\xi'}\cdot\eta')a+\alpha)$. Hence $(\lambda\lambda')\psi = (\xi\xi', \eta^{\xi'}\cdot\eta') = = (\xi, \eta)(\xi', \eta') = (\lambda\psi)(\lambda'\psi)$. Thus ψ is a homomorphism, and is clearly one-to-one.

If $(\xi, \eta)\in\Lambda(Q)wl\Phi$ satisfies (16), then λ defined by (17) has the property:

$$\begin{aligned}[\lambda(a, \alpha)](b, \beta) &= (\xi a, \eta a+\alpha)(b, \beta) = ((\xi a)b, [\xi a, b]+\eta a+\alpha+\beta)\\ &= (\xi(ab), \eta(ab)+[a, b]+\alpha+\beta) = \lambda(ab, [a, b]+\alpha+\beta)\\ &= \lambda[(a, \alpha)(b, \beta)].\end{aligned}$$

Thus $\lambda\in\Lambda(S)$ and furthermore $\lambda\psi=(\xi, \eta)$. Consequently

$$\Lambda(S)\psi=\{(\xi, \eta)|\xi \text{ and } \eta \text{ satisfy (16)}\}. \tag{19}$$

Next let $(a, \alpha)\in S$ and $\lambda_{(a,\alpha)}\psi=(\xi, \eta)$. Then for any $(b, \beta)\in S$,

$$\lambda_{(a,\alpha)}(b, \beta) = (a, \alpha)(b, \beta) = (ab, [a, b]+\alpha+\beta) \tag{20}$$

and on the other hand,

$$\lambda_{(a,\alpha)}(b, \beta) = (\xi b, \eta b+\beta). \tag{21}$$

Comparing (20) and (21), we obtain $\xi=\lambda_a$, $\eta b = [a, b]+\alpha$ for all $b\in Q$. Conversely, for the pair (λ_a, η) with $\eta b = [a, b]+\alpha$ for all $b\in Q$, we obtain

$$\eta b-\eta(bc) = [a, b]+\alpha-[a, bc]-\alpha = [b, c]-[ab, c] = [b, c]-[\lambda_a b, c]$$

using (6). Consequently

$$\Gamma(S)\psi = \{(\lambda_a, \eta)|\eta b = [a, b]+\alpha \text{ for some } \alpha\in\Phi\}.$$

Suppose now that Q is globally idempotent. Since $\Gamma(S)$ is a left ideal of $\Lambda(S)$, we have that $\Gamma(S)\psi$ is a left ideal of $\Lambda(S)\psi$. Now suppose that $(\sigma, \tau)\in\Lambda(S)wl\Phi$ has the property

$$(\sigma, \tau)(\xi, \eta)\in\Gamma(S)\psi \quad \text{for all} \quad (\xi, \eta)\in\Gamma(S)\psi.$$

It follows that (σ, τ) induces a left translation on $\Gamma(S)\psi$, and thus by the isomorphism ψ, there exists $\lambda\in\Lambda(S)$ such that $\lambda\psi$ and (σ, τ) have the same effect upon $\Gamma(S)\psi$. Writing $\lambda\psi=(\xi, \eta)$, we then have

$$(\xi, \eta)(\lambda_a, \theta)=(\sigma, \tau)(\lambda_a, \theta) \qquad ((\lambda_a, \theta)\in\Gamma(S)\psi).$$

Hence $\xi\lambda_a=\sigma\lambda_a$, $\eta^{\lambda_a}\cdot\theta = \tau^{\lambda_a}\cdot\theta$ so that $\xi\lambda_a x=\sigma\lambda_a x$, $\eta\lambda_a x+\theta x = \tau\lambda_a x+\theta x$ and thus $(\xi a)x=(\sigma a)x$, $\eta(ax)=\tau(ax)$ $(a, x\in Q)$. Right reductivity of Q implies $\xi=\sigma$, and $Q^2=Q$ implies $\eta=\tau$. Thus $(\xi, \eta)=(\sigma, \tau)$ which proves that $(\sigma, \tau)\in\Lambda(S)\psi$. Therefore $\Lambda(S)\psi$ is the largest subsemigroup of $\Lambda(Q)wl\Phi$ containing $\Gamma(S)\psi$ as a left ideal.

5.2 Lemma. *For a right reductive semigroup S, $i_{\Lambda(S)}(\Gamma(S))=\tilde{\Lambda}(S)$ and $\Gamma(S)$ is a densely embedded ideal of $\tilde{\Lambda}(S)$.*

Proof. Right reductivity of S implies that π_Λ is an isomorphism of $\Omega(S)$ onto $\tilde{\Lambda}(S)$ mapping $\Pi(S)$ onto $\Gamma(S)$. Since $\Pi(S)$ is a densely embedded ideal of $\Omega(S)$ by ([6], 1. 3. 5), $\Gamma(S)$ must be a densely embedded ideal of $\tilde{\Lambda}(S)$. In particular, $\tilde{\Lambda}(S)\subseteqq i_{\Lambda(S)}(\Gamma(S))$. If $\lambda\in i_{\Lambda(S)}(\Gamma(S))$, then for every $a\in S$, there exists $a'\in S$ such that $\lambda_a\lambda=\lambda_{a'}$, where a' is unique by right reductivity of S. Define ϱ on S by the requirement $\lambda_a\lambda=\lambda_{a\varrho}$ $(a\in S)$. Then

$$\lambda_{(ab)\varrho}=\lambda_{ab}\lambda=\lambda_a(\lambda_b\lambda)=\lambda_a\lambda_{b\varrho}=\lambda_{a(b\varrho)},$$

$$\lambda_{(a\varrho)b}=(\lambda_{a\varrho})\lambda_b=(\lambda_a\lambda)\lambda_b=\lambda_a(\lambda\lambda_b)=\lambda_a\lambda_{\lambda b}=\lambda_{a(\lambda b)}$$

so that $(\lambda, \varrho)\in\Omega(S)$. Consequently $\lambda\in\tilde{\Lambda}(S)$ proving that $i_{\Lambda(S)}(\Gamma(S))\subseteqq\tilde{\Lambda}(S)$.

5.3 Corollary. *For a right reductive semigroup Q, isomorphism* (18) *induces an embedding of S into $\Gamma(Q)wl\Phi$, and if $Q^2=Q$, it also induces an l-dense and dense embedding of S into $\Lambda(Q)wl\Phi$.*

Proof. It suffices to compose the isomorphism $a\to\lambda_a$ with ψ and apply 5. 1, 5. 2 and ([6], 1. 3. 5 and 1. 10. 2).

5.4 Corollary. *Every right reductive semigroup S for which $GZ(S)$ acts simply transitively on each σ-class can be embedded into $\Gamma(S/\sigma)wlGZ(S)$. If S is also globally idempotent, then S can also be l-densely and densely embedded into $\Lambda(S/\sigma)wlGZ(S)$.*

Proof. Apply 4. 3 and 5. 3.

6. Cancellative semigroups. For the class of cancellative semigroups, we are able to prove much stronger statements concerning σ_S than in the general case. Throughout this section S denotes an arbitrary cancellative semigroup.

6.1 Proposition. *For any $a, b \in S$, we have $a\sigma b$ if and only if*

$$aS=bS, \quad Sa=Sb, \quad axb=bxa \quad \textit{for all} \quad x \in S. \tag{22}$$

Proof. First let $a=\lambda b$ where $\lambda \in GZ(S)$. For any $x \in S$, we obtain $ax=(\lambda b)x=$ $=b(\lambda x) \in bS$, $xa=x(\lambda b)=(\lambda x)b \in Sb$ proving $aS \subseteqq bS$ and $Sa \subseteqq Sb$. By symmetry, we also have $bS \subseteqq aS$ and $Sb \subseteqq Sa$. Further, $axb=ax(\lambda a)=(\lambda a)xa=bxa$, and thus the pair a, b satisfies (22).

Conversely, let $a, b \in S$ satisfy (22). For every $x \in S$ there exists a unique $y \in S$ such that $ax=by$. We then define a function λ on S by the requirement $ax=b(\lambda x)$ $(x \in S)$. Similarly define λ' by $bx=a(\lambda' x)$ $(x \in S)$. Then $ax=b(\lambda x)=a(\lambda' \lambda x)$ so that $x=\lambda' \lambda x$ and similarly $x=\lambda\lambda' x$ for all $x \in S$, which shows that λ is invertible. Furthermore, λ is obviously a left translation of S. Analogously define ϱ on S by $xa=$ $=(x\varrho)b$; a dual proof shows that ϱ is an invertible right translation on S. For any $x \in S$, we also have $b(\lambda x)b=axb=bxa=b(x\varrho)b$ so that $\lambda x=x\varrho$. But then part b) of 3.3 implies that $(\lambda, \varrho) \in C\Omega(S)$, and hence $\lambda \in GZ(S)$. Finally $ax=b(\lambda x)$ implies $ax=(\lambda b)x$ so that $a=\lambda b$.

6.2 Corollary. *Let S be a cancellative semigroup without idempotents. Then there exists a nontrivial group G and a cancellative semigroup V which is an ideal extension of S by G^0 such that $G \subseteqq C(V)$ if and only if there exist distinct elements a and b of S for which $aS=bS$, $Sa=Sb$, $axb=bxa$ for all $x \in S$.*

Proof. The last condition is equivalent to the statement that σ_S is not the equality relation on S, which is in turn equivalent to the assertion that $GZ(S)$ is nontrivial. Now if $V = S \cup G$ is an extension of S described above, then V is a dense extension of S and the canonical homomorphism $G \to \Omega(S)$ provides an isomorphism of G into $GC\Omega(S)$ (see [7] for a general discussion). Hence if G is nontrivial, so is $GC\Omega(S)$ and thus also $GZ(S)$. Conversely, if $GZ(S)$ is nontrivial, then $V = S \cup GC\Omega(S)$ with the identification of S and $\Pi(S)$, provides an extension of S of the desired form.

For the case of a commutative cancellative S, this corollary reduces to ([8], Theorem 4.4). The next lemma is also of independent interest.

6.3 Lemma. *A dense extension of a left cancellative right reductive semigroup is left cancellative and right reductive.*

Proof. Let V be a dense extension of a left cancellative right reductive semigroup S. Let $a, b \in V$ and suppose that $as=bs$ for all $s \in S$. In particular $(ta)s=$

$=(tb)s$ for all $t, s \in S$ where $ta, tb \in S$, so by right reductivity of S we obtain $ta=tb$. Consequently $as=bs$, $sa=sb$ for all $s \in S$, which by ([7], Theorem 3. 7) implies $a=b$ since V is a dense extension of S. In particular, V is right reductive. Suppose next that $ca=cb$ for some $a, b, c \in V$. Then $(sc)(at)=(sc)(bt)$ for all $s, t \in S$, where $sc, at, bt \in S$. Hence the left cancellation in S yields $at=bt$ for all $t \in S$. But then $a=b$ as we have seen above. Thus V is left cancellative.

6. 4 Corollary. *If S is cancellative, so is $\Omega(S)$.*

Proof. This follows from 6. 3, its dual, and the fact that $\Omega(S)$ is a (maximal) dense extension of $\Pi(S) \cong S$ by ([6], 1. 3. 5).

Let S be a cancellative semigroup, and for every $a \in S$ denote by a^* the σ-class containing a. Let $a\sigma b$ and $(\lambda, \varrho) \in \Omega(S)$. Then $a=\varphi b$ for some $\varphi \in GZ(S)$ which implies $\lambda a=\lambda\varphi b=\varphi(\lambda b)$ and $a\varrho=(\varphi b)\varrho=\varphi(b\varrho)$, so that $\lambda a\sigma\lambda b$ and $a\varrho\sigma b\varrho$. This makes it possible to define the function θ below.

6. 5 Theorem. *For a cancellative semigroup S, define a function θ by:*

$$\theta\colon (\lambda, \varrho) \to (\bar{\lambda}, \bar{\varrho}) \quad ((\lambda, \varrho) \in \Omega(S)) \tag{23}$$

where

$$\bar{\lambda}a^*=(\lambda a)^*,\ a^*\bar{\varrho}=(a\varrho)^* \quad (a \in S). \tag{24}$$

Then θ is a homomorphism of $\Omega(S)$ into $\Omega(S/\sigma_S)$ and $\ker\theta=\sigma_{\Omega(S)}$. *Moreover $\Omega(S/\sigma_S)$ is cancellative.*

Proof. The discussion before the theorem shows that both $\bar{\lambda}$ and $\bar{\varrho}$ are single-valued. For any $a, b \in S$, we obtain

$$(\bar{\lambda}a^*)b^*=(\lambda a)^*b^*=[(\lambda a)b]^*=[\lambda(ab)]^*=\bar{\lambda}(ab)^*=\bar{\lambda}(a^*b^*)$$

so $\bar{\lambda} \in \Lambda(S/\sigma)$, and dually $\bar{\varrho} \in P(S/\sigma)$; that $\bar{\lambda}$ and $\bar{\varrho}$ are linked is verified in a similar manner. Thus θ maps $\Omega(S)$ into $\Omega(S/\sigma)$. For $(\lambda, \varrho), (\varphi, \psi) \in \Omega(S)$ and $a \in S$, we have

$$(\bar{\lambda}\bar{\varphi})a^*=\bar{\lambda}(\bar{\varphi}a^*)=\bar{\lambda}(\varphi a)^*=(\lambda\varphi a)^*=\overline{\lambda\varphi}\,a^*$$

so that $\bar{\lambda}\bar{\varphi}=\overline{\lambda\varphi}$ and dually $\bar{\varrho}\bar{\psi}=\overline{\varrho\psi}$, showing that θ is a homomorphism.

Next let $(\lambda, \varrho), (\varphi, \psi) \in \Omega(S)$. Then $(\lambda, \varrho)\theta=(\varphi, \psi)\theta$ is successively equivalent to $\bar{\lambda}=\bar{\varphi}$, $\bar{\varrho}=\bar{\psi}$, and to

$$(\lambda a)^*=(\varphi a)^*, \quad (a\varrho)^*=(a\psi)^* \qquad (a \in S),$$

and to

$$\lambda a\,\sigma\,\varphi a, \quad a\varrho\,\sigma\,a\psi \qquad (a \in S). \tag{25}$$

Suppose that $\lambda a\sigma\varphi a$ for all $a \in S$. By 6. 1, we have

$$(\lambda a)S=(\varphi a)S, \quad S(\lambda a)=S(\varphi a), \quad (\lambda a)x(\varphi a)=(\varphi a)x(\lambda a) \qquad (x \in S).$$

For a fixed $a\in S$, as in the proof of 6. 1, we may define γ and δ by:

$$(\lambda a)x=(\varphi a)(\gamma x),\quad x(\lambda a)=(x\delta)(\varphi a)\qquad (x\in S).$$

It follows as in the proof of 6. 1 that $(\gamma,\delta)\in CG\Omega(S)$ and $\lambda a=\gamma\varphi a$. Further $(a\varrho)a=$ $=a(\lambda a)=a(\gamma\varphi a)=(a\delta\psi)a$ so $a\varrho=a\delta\psi$. For any $b\in S$ we then have $a(\lambda b)=(a\varrho)b=$ $=(a\delta\psi)b=a(\gamma\varphi b)$ and thus $\lambda b=\gamma\varphi b$. As in the preceding step, this also implies $b\varrho=b\delta\psi$. Consequently $(\lambda,\varrho)=(\gamma,\delta)(\varphi,\psi)$ with $(\gamma,\delta)\in GC\Omega(S)$ and hence

$$(\lambda,\varrho)\sigma_{\Omega(S)}(\varphi,\psi).$$

Note that we have used only the first half of (25) and for a single a. Hence $\ker\theta\subseteq$ $\subseteq\sigma_{\Omega(S)}$.

Conversely, suppose that $(\lambda,\varrho)\sigma_{\Omega(S)}(\varphi,\psi)$. Then for some $(\gamma,\delta)\in GC\Omega(S)$, we have $(\lambda,\varrho)=(\gamma,\delta)(\varphi,\psi)$, and thus $\lambda=\gamma\varphi$, $\varrho=\delta\psi$. Thus for every $a\in S$, we obtain $\lambda a=\gamma(\varphi a)$ and $a\varrho=(a\delta)\psi=(\gamma a)\psi=\gamma(a\psi)$, and hence (25) holds since $\gamma\in GZ(S)$. We have seen above that this is equivalent to $(\lambda,\varrho)\theta=(\varphi,\psi)\theta$. Consequently $\sigma_{\Omega(S)}\subseteq$ $\subseteq\ker\theta$, and the equality prevails.

Suppose that $a^*c^*=b^*c^*$. Then $ac\,\sigma\,bc$ and thus $ac=\lambda(bc)$ for some $\lambda\in GZ(S)$. But then $ac=(\lambda b)c$ so that $a=\lambda b$ and thus $a^*=b^*$. It follows that right cancellation in S implies right cancellation in S/σ. By symmetry, we conclude that S/σ is cancellative, which by 6. 4 implies that $\Omega(S/\sigma)$ is cancellative.

The next result shows that for every element b of a cancellative semigroup S we can define a new multiplication on a subset of S in such a way as to make it a semigroup isomorphic with $Z(S)$ and for which b acts as the identity element. The group of units of this semigroup, as a set, coincides with the σ_S-class of b.

6. 6 Theorem. *Let S be a cancellative semigroup. For any $b\in S$, let*

$$\Sigma_b=\{a\in S\,|\,aS\subseteq bS,\quad Sa\subseteq Sb,\quad axb=bxa\quad \text{for all}\quad x\in S\},$$

and on Σ_b define multiplications $*$ and $\circ$ by the formulae:

$$aa'=b(a*a')=(a\circ a')b.$$

Then Σ_b is closed under $*$, the two multiplications coincide, and the mapping ψ defined by $\psi:\lambda\to\lambda b\ (\lambda\in Z(S))$ is an isomorphism of $Z(S)$ onto $(\Sigma_b,*)$. In $(\Sigma_b,*)$, b is the identity element and $(b^*,*)=G(\Sigma_b,*)\cong GZ(S)$.

Proof. For $\lambda\in Z(S)$ and any $x\in S$, we obtain $(\lambda b)x=b(\lambda x)\in bS$, $x(\lambda b)=$ $=(\lambda x)b\in Sb$, $(\lambda b)xb=bx(\lambda b)$ which shows that $\lambda b\in\Sigma_b$. If $a\in\Sigma_b$, then similarly as in the second part of the proof of 6. 1, we may show that λ defined by $ax=b(\lambda x)$ $(x\in S)$ has the properties $\lambda\in Z(S)$ and $a=\lambda b$. Thus ψ maps $Z(S)$ onto Σ_b. If $\lambda,\lambda'\in$

$\in Z(S)$ and $\lambda b = \lambda' b$, then for any $x \in S$, $(\lambda x)b = x(\lambda b) = x(\lambda' b) = (\lambda' x)b$ so that $\lambda = \lambda'$ and ψ is one-to-one. For $\lambda, \lambda' \in Z(S)$, we further have

$$b[(\lambda b) * (\lambda' b)] = (\lambda b)(\lambda' b) = b(\lambda\lambda' b)$$

so that $(\lambda\psi) * (\lambda'\psi) = (\lambda\lambda')\psi$ showing that ψ is a homomorphism. Therefore ψ is an isomorphism of $Z(S)$ onto Σ_b, which in particular implies that Σ_b is closed under $*$. For any $a, a' \in \Sigma_b$, we also have

$$b(a * a')b = aa'b = ba'a = b(a' \circ a)b$$

and hence $a * a' = a' \circ a$. But the isomorphism $Z(S) \cong (\Sigma_b, *)$ shows that $a * a' = a' * a$, and we conclude that $*$ and $\circ$ coincide. It is immediate that b is the identity of $(\Sigma_b, *)$, and a comparison with the definition of σ_S quickly shows that the last assertion of the theorem is correct.

Example. Consider the set $S = \{(a, b) | 0 < a < 1, b \text{ real}\}$ under the multiplication $(a, b)(c, d) = (ac, bc + d)$. The mapping

$$(a, b) \to \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$$

is easily seen to be an isomorphism of S into the multiplicative group of 2×2 non-singular matrices over reals. Thus S is a cancellative semigroup. A straightforward calculation shows that the translational hull of S can be identified with the semigroup

$$T = \{(a, b) | 0 < a \leqq 1, \ b \text{ real}\}$$

with elements of T acting on elements of S and multiplying among themselves by the same rule as in S. One further verifies easily that

(i) $G(T) = CG(T) = \{(1, b) | b \text{ real}\}$

(ii) $GZ(S) = Z(S) \cong GC(T) = C(T) = \{(1, 0)\}$.

In particular, $GC\Omega(S)$ is trivial while $CG\Omega(S) = G\Omega(S) = C\Omega(S)$ is isomorphic to the group of additive real numbers.

In [4] (Chapitre II, § 4) Dubreil defines an "inner automorphism" of a cancellative semigroup S as follows. For $a \in S$ such that $aS = Sa$, define α_a and β_a by the formulae: $ax = (\alpha_a x)a$, $xa = a(\beta_a x)$ $(s \in S)$. Then α_a and β_a are called inner automorphisms of the first and second category, respectively. It is easy to see that for any cancellative semigroup S and $(\lambda, \varrho) \in G\Omega(S)$, $a \in S$, we have $aS = Sa$ and $\alpha_a = \delta_{(\lambda, \varrho)}$ if and only if $\lambda a \in C(S)$. In the above example,

(iii) $(a, b)S = S(a, b)$ for all $(a, b) \in S$,

(iv) $C(S) = \emptyset$.

Hence S has inner automorphisms of the first (and thus also of the second) category but none is a generalized inner automorphism in our sense. In the case of groups, however, both of these notions reduce to inner automorphisms. For further properties of this type of example, see ([1], § 2. 1, exerc. 9).

7. Commutative cancellative semigroups. For a semigroup S of this class, we can give much more precise and complete information concerning the representation of S as a Schreier extension of $GZ(S)$ by S/σ_S. We start with some auxiliary results.

7. 1 Lemma. *A dense extension V of a commutative reductive semigroup S is commutative.*

Proof. Let $a, b \in V$; then for any $s, t \in S$, we have $s(ab)t=(sa)(bt)=(bt)(sa)= =b(sa)t=bs(at)=(bat)s=s(ba)t$ which by reductivity in S yields $sab=sba$. Since this holds for all $s \in S$, ([7], Theorem 3. 7) implies $ab=ba$ by density of the extension.

In fact, the above V is also reductive, which we will not need here.

7. 2 Corollary. *If S is commutative and cancellative, so is $\Omega(S)$.*

Proof. This follows from 7. 1 and ([6], 1. 3. 5).

7. 3. Lemma. *For any commutative semigroup S we have $Z(S)=\tilde{\Lambda}(S)=\Lambda(S)$.*

Proof. If $\lambda \in \Lambda(S)$, then letting $s\varrho=\lambda s$ $(s \in S)$, we obtain $(\lambda, \varrho) \in \Omega(S)$, which shows that $\Lambda(S) \subseteqq \tilde{\Lambda}(S)$. The inclusion $\Lambda(S) \subseteqq Z(S)$ follows immediately from commutativity.

We infer that for a commutative reductive semigroup, the projection π_Λ furnishes an isomorphism of $\Omega(S)$ onto $\Lambda(S)$, and both of these are commutative. In order to simplify our statements, we introduce the following concept.

7. 4. Definition. A semigroup S is *basic* if S is commutative, cancellative, and $aS=bS$ implies $a=b$.

In view of commutativity and 6. 1, the last condition is equivalent to σ_S being the equality relation. Note that in general $\sigma_{S/\sigma}$ need not be the equality relation, it suffices to take a group G for which $G/C(G)$ has a nontrivial center. For the semigroups under consideration, we have

7. 5. Proposition. *Let S be a commutative cancellative semigroup. Then σ_S is the smallest congruence τ on S for which S/τ is basic.*

Proof. At the end of the proof of 6. 5, we have seen that cancellation in S implies cancellation in S/σ_S. Next suppose that $a^* \sigma_{S^*} b^*$ where $a \to a^*$ is the canonical homomorphism of S onto $S/\sigma=S^*$. By 6. 1, we have $a^* S^*=b^* S^*$. Thus for every $x \in S$ there exists $y \in S$ such that $a^* x^*=b^* y^*$. But then $(ax)^*=(by)^*$ which implies $axS=byS$. In particular, there exists $z \in S$ for which $axy=byz$, and thus $ax=bz$.

This shows that $aS \subseteqq bS$; by symmetry we conclude that $aS=bS$, i.e., $a\sigma_S b$. We have proved that $a^* \sigma_{S^*} b^*$ implies $a^*=b^*$, and S/σ_S is basic.

Next let τ be any congruence on S for which S/τ is basic, and let $a \to \hat{a}$ be the canonical homomorphism of S onto $S/\tau=\hat{S}$. If $a\sigma_S b$, then $aS=bS$ so $\hat{a}\hat{S}=\hat{b}\hat{S}$, and since $\hat{S}$ is basic, it follows that $\hat{a}=\hat{b}$. Hence $a\tau b$ proving that $\sigma_S \subseteqq \tau$.

We come now to the principal theorem of this section. It is the culmination of the effort to construct commutative cancellative semigroups out of commutative cancellative semigroups having some special properties and using $GZ(S)$ and S/σ.

7.6 Theorem. *Let Q be a basic semigroup, Φ be an abelian group, $S=(Q, \Phi, [\;,\;])$ be a Schreier extension of Φ by Q, and suppose that*

$$[a, b]=[b, a] \qquad (a, b \in Q). \tag{26}$$

Then S is a commutative cancellative semigroup for which $S/\sigma_S \cong Q$, and $GZ(S) \cong \Phi$. Conversely, every commutative cancellative semigroup S is isomorphic to $(Q, \Phi, [\;,\;])$ for some basic semigroup Q, abelian group Φ, and a function $[\;,\;]$ satisfying (6) *and* (26).

Proof. Let S be as in the first part of the theorem. A simple calculation shows that S is both commutative and cancellative. The mapping $\psi:(a, \alpha) \to a$ for all $(a, \alpha) \in S$, is obviously a homomorphism of S onto Q. Let $(a, \alpha)\sigma_S(b, \beta)$. Then $(a, \alpha)S=(b, \beta)S$ and hence for any $c \in Q$ there exists $(d, \gamma) \in S$ such that $(a, \alpha)(c, \alpha)= =(b, \beta)(d, \gamma)$. But then $ac=bd$ which implies $aQ \subseteqq bQ$. By symmetry, we also have $bQ \subseteqq aQ$ so that $aQ=bQ$. Since Q is basic, we infer that $a=b$, which in turn implies that $(a, \alpha)\psi=(b, \beta)\psi$. Consequently $\sigma_S \subseteqq \ker \psi$. To prove the converse, by symmetry, it suffices to show that $(a, \alpha)S \subseteqq (a, \beta)S$ for any $a \in Q$, $\alpha, \beta \in \Phi$. Indeed, for any $(c, \gamma) \in S$, we obtain

$$(a, \alpha)(c, \gamma) = (ac, [a, c]+\alpha+\gamma) = (a, \beta)(c, \alpha+\gamma-\beta) \in (a, \beta)S.$$

Thus $\sigma_S=\ker \psi$ and $S/\sigma_S \cong Q$.

We prove next the second isomorphism. Fix $b \in Q$ and let 0 be the identity element of Φ. For any $\lambda \in GZ(S)$, $\lambda(b, 0)$ is of the form (b, β) for some $\beta \in \Phi$, for as we have proved above, $(a, \alpha)\sigma_S(b, \beta)$ if and only if $a=b$, i.e., λ must preserve the first entry. Hence we may define a function χ from $GZ(S)$ into Φ as follows:

$$\lambda(b, 0)=(b, \lambda\chi) \qquad (\lambda \in GZ(S)).$$

For $\lambda, \lambda' \in GZ(S)$, we have

$$(b, 0)[\lambda\lambda'(b, 0)] = \lambda(b, 0)\lambda'(b, 0) = (b, \lambda\chi)(b, \lambda'\chi) = (b^2, [b, b]+\lambda\chi+\lambda'\chi),$$

$$(b, 0)[\lambda\lambda'(b, 0)] = (b, 0)(b, (\lambda\lambda')\chi) = (b^2, [b, b]+(\lambda\lambda')\chi),$$

which implies $\lambda\chi+\lambda'\chi=(\lambda\lambda')\chi$, i.e., χ is a homomorphism. If $\lambda\chi=\lambda'\chi$, then $\lambda(b, 0)=$ $=\lambda'(b, 0)$, and hence for any $(a, \alpha)\in S$,

$$[\lambda(a, \alpha)](b, 0)=[\lambda(b, 0)](a, \alpha)=[\lambda'(b, 0)](a, \alpha)=[\lambda(a, \alpha)](b, 0)$$

which shows that $\lambda=\lambda'$, so χ is one-to-one. Next let $\beta\in\Phi$. From what we have seen above, it follows that $(b, \beta)S=(b, 0)S$. It then follows easily that the function λ defined by the formula:

$$(b, \beta)(a, \alpha)=(b, 0)(\lambda(a, \alpha)) \qquad ((a, \alpha)\in S)$$

has the properties $\lambda\in GZ(S)$ and $\lambda(b, 0)=(b, \beta)$. Hence $\lambda\chi=\beta$ proving that χ maps $GZ(S)$ onto Φ. Therefore $GZ(S)\cong\Phi$.

The converse follows immediately from 4. 3 and 7. 5, formula (26) follows from commutativity of S.

7. 7 Corollary. *Let* $S=(Q, \Phi, [\ ,\ ])$ *and* $S'=(Q', \Phi', [\ ,\ ]')$ *where* $[\ ,\ ]$ *and* $[\ ,\ ]'$ *satisfy* (26), Q *and* Q' *are basic. Then the conditions in* 4. 4 *are necessary and sufficient for isomorphism of* S *and* S'.

Proof. This follows from 7. 6 and the proof of 4. 4.

7. 8 Corollary. *Every commutative cancellative semigroup* S *can be embedded into* $\Gamma(Q)wl\,\Phi$ *and l-densely and densely embedded into* $\Lambda(Q)wl\,\Phi$ *for some basic semigroup* Q *and an abelian group* Φ.

Proof. This follows from 7. 6 and 5. 4.

As an example, we compute all functions $[\ ,\ ]$ for two very simple basic semigroups.

7. 9 Proposition. *Let* N *be the additive semigroup of positive integers and* Φ *be an abelian group. For a sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *of elements of* Φ, *define a function* $[\ ,\ ]$ *by:*

$$[m, 1]=\alpha_m,\ [m, n]=\alpha_m+\sum_{i=1}^{n-1}(\alpha_{m+i}-\alpha_i) \quad (m\geqq 1, n\geqq 2). \tag{27}$$

Then $[\ ,\ ]$ *satisfies both* (6) *and* (26). *Conversely, every function* $[\ ,\ ]$ *from* N *into* Φ *satisfying* (6) *can be so obtained.*

Proof. For a function $[\ ,\ ]$ defined by (27), it is routine to verify that it satisfies (6) and (26). Conversely, let $[\ ,\ ]$ be a function from N into Φ satisfying (6). We let $\alpha_m=[m, 1]$ $(m\in N)$, then the second part of (27) can be written as

$$[m, n]=[m, 1]+\sum_{i=1}^{n-1}([m+i, 1]-[i, 1]) \quad (m\geqq 1, n\geqq 2).$$

The proof of this relation is by induction on n for a fixed m. The case of $n=1$ is trivial. Suppose the formula correct for n. By (6), we have

$$[m, n]+[m+n, 1] = [m, n+1]+[n, 1]$$

which implies

$$[m, n+1] = [m, n]+[m+n, 1]-[n, 1] =$$
$$= [m, 1]+\sum_{i=1}^{n-1}([m+i, 1]-[i, 1])+[m+n, 1]-[n, 1] =$$
$$= [m, 1]+\sum_{i=1}^{n}([m+i, 1]-[i, 1])$$

as required.

7.10 Proposition. *Let N° be the additive semigroup of nonnegative integers and Φ be an abelian group. For any sequence $\{\alpha_n\}_{n=0}^{\infty}$ of elements of Φ define a function $[\ ,\ ]$ by: $[m, 0]=[0, m]=\alpha_0$ for $m\geqq 0$ and* (27) *for the remaining values. Then $[\ ,\ ]$ satisfies both* (6) *and* (26). *Conversely, every function $[\ ,\ ]$ from N° into Φ satisfying* (6) *can be so obtained.*

Proof. A proof using 7.9 and considering the extra elements of the form $[m, 0]$, $[0, m]$ is straightforward and is omitted.

7.11 Corollary. *If S is a cancellative semigroup such that S/σ_S is isomorphic to either N or N°, then S is commutative.*

Proof. By 4.3, S is isomorphic to a Schreier extension of the abelian group $GZ(S)$ and N or N°. Now 7.9 and 7.10 imply that the corresponding function $[\ ,\ ]$ automatically satisfies (26) which implies the commutativity of the Schreier extension and thus also of S.

Since 7.9 and 7.10 yield all functions $[\ ,\ ]$, we are able to construct all Schreier extensions of Φ by N or N°. Even though 7.7 gives necessary and sufficient conditions for isomorphism of such extensions, we are unable to tell which sequences will yield isomorphic semigroups.

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