Isomorphism types of objects in categories determined by numbers of morphisms

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Consider a category $\mathfrak{A}$ and, for an object a of $\mathfrak{A}$ the family $(|\mathfrak{A}(x, a)|)_{x \in \mathrm{obj}\, \mathfrak{A}}$ of numbers of morphisms from all objects of $\mathfrak{A}$ into a. In [3], L. LOVÁSZ showed that the category of finite sets with systems of k-nary relations and the category of finite k-partite structures have the property that this family of numbers determines the isomorphism type of a, and used this fact to prove his product cancellation laws. Later on, in [2], he developed another method enabling him to prove the cancellation laws for general categories.

In the present paper a sufficient condition is given (Theorem 2. 2) on a category $\mathfrak{A}$ to have the property described above (cf. Definition 1. 7 below). Moreover, we prove that for every category such that there is only a finite number of morphisms between any two objects there is a full product preserving embedding into a category with that property (Theorems 2. 5 and 3. 3), which gives an information on the structure of product semigroups of isomorphism types (Corollary 3. 4). Also, a proof is given of one of LOVÁSZ' cancellation laws which is not an immediate consequence of the property above, for general categories, but based on the original idea from [3]. In fact, the proof of Theorem 2. 2 also exploits the idea from [3] in general categorial language.

§ 1. Preliminaries

1. 1. Definitions and notation. The category of sets (finite sets) and their mappings will be denoted by Set (Set f). If $\mathfrak{A}$ is a category, the symbol $\mathfrak{A}$ is also used for the natural functor $\mathfrak{A}^{op} \times \mathfrak{A} \to$ Set (i.e., $\mathfrak{A}(a, b)$ is the set of morphisms from a into b in $\mathfrak{A}$, $\mathfrak{A}(\varphi, \psi)(\alpha) = \psi\alpha\varphi$).

A morphism $\varepsilon : a \to b$ in $\mathfrak{A}$ is said to be a quotient in $\mathfrak{A}$, if 1_b is its image. The set of quotients from a into b is denoted by Quo (a, b). The set of monomorphisms and isomorphisms from a into b is denoted by Mono (a, b) and Iso (a, b), respectively.

If M is a set, $|M|$ is its cardinality.

If A is a small category and $\mathfrak{B}$ a category, $\mathfrak{B}^A$ designates the category of all functors $A \to \mathfrak{B}$ and their transformations.

1.2. Remark. Obviously, the quotients which are epimorphisms are exactly the well known extremal epimorphisms.

1.3. Definition. A category $\mathfrak{A}$ is said to be *quasifinite* if every $\mathfrak{A}(a, b)$ is finite. On the other hand, the expression *locally finite* will be preserved for categories with only a finite number of non-equivalent monomorphisms into each object (in accordance with the common use of "locally small" and "colocally small").

1.4. The following well known statements will be used without mentioning (they are either explicitely in textbooks or quite trivial to prove):

1) If $\varphi = \mu \cdot \varepsilon$ and μ is an image of φ, then ε is a quotient.

2) If $\varphi = \mu \cdot \varepsilon$ with a monomorphism μ and a quotient ε, then μ is an image of φ.

3) If $\mathfrak{A}$ has equalizers and ε is a quotient in $\mathfrak{A}$ then ε is an epimorphism.

4) A locally finite category with intersections has images.

5) If $\mathfrak{B}$ is complete (cocomplete, finitely complete, finitely cocomplete) then every $\mathfrak{B}^A$ is. The evaluation functors $\mathfrak{B}^A \to \mathfrak{B}$ preserve limits and colimits, consequently, monomorphisms and epimorphisms.

6) If A is finite and $\mathfrak{B}$ locally finite then $\mathfrak{B}^A$ is locally finite.

1.5. Lemma (coincides with Lemma 1 in [2]). *Let $\mathfrak{A}(a, a)$, $\mathfrak{A}(b, b)$ be finite, let $\mu : a \to b$, $\nu : b \to a$ be monomorphisms (epimorphisms, resp.). Then μ and ν are isomorphisms.*

Proof. There are integers $n \geqq 0$, $k > 0$ with $(\nu\mu)^{n+k} = (\nu\mu)^n$. Since $\nu\mu$ is a monomorphism, hence, $\nu((\mu\nu)^{k-1}\mu) = 1$. Thus, ν is a retraction and a monomorphism, hence an isomorphism.

1.6. Lemma. *Let $\mathfrak{A}$ be locally finite quasifinite, let x, a, b be its objects, T a system of objects of $\mathfrak{A}$ containing exactly one representant of each isomorphism class. Then there are only finitely many $\varphi : x \to d$ such that $d \in T$ and $|\mathrm{Mono}\,(d, a)| \neq$ $\neq |\mathrm{Mono}\,(d, b)|$.*

Proof. If $|\mathrm{Mono}\,(d, a)| \neq |\mathrm{Mono}\,(d, b)|$, at least one of them is not zero. Since no two $d, t \in T$ are isomorphic and $\mathfrak{A}$ is locally finite, there are only finitely many such d. Since $\mathfrak{A}$ is quasifinite, the statement follows.

1.7. Definition. A semifinite category $\mathfrak{A}$ is said to be *combinatorial* if $(\forall x\, |\mathfrak{A}(x, a)| = |\mathfrak{A}(x, b)|)$ implies that a is isomorphic to b.

§ 2. A sufficient condition for a category to be combinatorial. Consequences

2. 1. Theorem. *Let a, b be objects of $\mathfrak{A}$ such that every $\varphi: a \to b$ has an image. Let T be a system of objects of $\mathfrak{A}$ containing exactly one representant of each isomorphism class. Then*

$$|\mathfrak{A}(a, b)| = \sum_{t \in T} |\mathrm{Iso}\,(t, t)|^{-1} |\mathrm{Quo}\,(a, t)| \cdot |\mathrm{Mono}\,(t, b)|.$$

Proof. Define an equivalence relation e_t on $R_t = \mathrm{Quo}\,(a, t) \times \mathrm{Mono}\,(t, b)$ by $(\varepsilon, \mu) e_t (\varepsilon', \mu')$ iff $\mu\varepsilon = \mu'\varepsilon'$. Thus, $|\mathfrak{A}(a, b)| = \sum_T |R_t/e_t|$. On the other hand (see 1. 4. 2)) obviously

$$|R_t/e_t| = |\mathrm{Iso}\,(t, t)|^{-1} \cdot |\mathrm{Quo}\,(a, t)| \cdot |\mathrm{Mono}\,(t, b)|.$$

2. 2. Theorem. *Let $\mathfrak{A}$ be a locally finite quasifinite category with images such that every quotient is an epimorphism. Then it is combinatorial.*

Proof. Let $|\mathfrak{A}(t, a)| = |\mathfrak{A}(t, b)|$ for every $t \in T$ (T from 2. 1). We shall prove that then $|\mathrm{Mono}\,(t, a)| = |\mathrm{Mono}\,(t, b)|$ for every $t \in T$ which shall prove the statement by 1. 5. Thus, let $|\mathrm{Mono}\,(t, a)| \neq |\mathrm{Mono}\,(t, b)|$. By 2. 1,

$$0 = \sum_{d \neq t} \frac{|\mathrm{Quo}\,(t, d)|}{|\mathrm{Iso}\,(d, d)|} \cdot \big(|\mathrm{Mono}\,(d, a)| - |\mathrm{Mono}\,(d, b)|\big) + \big(|\mathrm{Mono}\,(t, a)| - |\mathrm{Mono}\,(t, b)|\big).$$

($\mathrm{Quo}\,(t, t) = \mathrm{Iso}\,(t, t)$ by 1. 5 and the assumption on quotients). Thus, there is a non-isomorphic quotient $\varepsilon(t): t \to \bar{t}$ such that $|\mathrm{Mono}\,(\bar{t}, a)| \neq |\mathrm{Mono}\,(\bar{t}, b)|$. Put $t_0 = t$, $t_{n+1} = \bar{t}_n$, $\varepsilon_n = \varepsilon(t_n)$. By 1. 6 there are natural k and $n > k$ with $\varepsilon_k \cdot \varepsilon_{k-1} \cdot \dots \cdot \varepsilon_0 = \varepsilon_n \cdot \varepsilon_{n-1} \cdot \dots \cdot \varepsilon_0$. By the assumption on quotients, thus, $\varepsilon_n \cdot \dots \cdot \varepsilon_{k+1} = 1$. Hence ε_{k+1} is a coretraction and an epimorphism, so that it is an isomorphism, which is a contradiction.

2. 3. Corollary. *Every locally finite quasifinite finitely complete category is combinatorial.*

Proof. Since it is locally finite and has finite intersections, it has images (see e.g. [4]). Since it has equalizers, the quotients are epimorphisms.

2. 4. Remark. Theorem 2. 2 holds also under dual conditions. It suffices to use an analogon of 2. 1 with coimage decomposition of the morphisms, epimorphisms instead of quotients and subobjects (i.e. morphisms with the identity for a coimage) instead of monomorphisms.

2. 5. Theorem. *For every finite category A there is a full product preserving embedding $\Phi: A \to C$ with C combinatorial.*

Proof. The Yoneda embedding $Y: A \to \mathrm{Setf}^{A^{op}}$ (given by $Y(a)(b) = A(b, a)$, $Y(\varphi)^b = A(\varphi, 1_b)$) is a full product preserving embedding. By 2. 3 and 1. 4. 5) 6), $\mathrm{Setf}^{A^{op}}$ is combinatorial.

2. 6. As an immediate consequence of 2. 5 and of the trivial fact that $|\mathfrak{A}(x, a \times b)| = |\mathfrak{A}(x \cdot a)| \cdot |\mathfrak{A}(x, b)|$ we obtain (exactly like the analogous statements for special categories in [3]) the general LovÁsz' cancellation laws from [2] (we reformulate them slightly):

1) Let $n \geqq 1$ and a^n be isomorphic to b^n in a category $\mathfrak{A}$. If there is only a finite number of morphisms between a, b, a^n then a is isomorphic to b.

2) Let $a \times c$ be isomorphic to $b \times c$ in $\mathfrak{A}$, let $\mathfrak{A}(d, c)$ be non-void. If there is only a finite number of morphisms between a, b, c, $a \times c$, $a \times d$, $b \times d$ then $a \times d$ is isomorphic to $b \times d$.

By duality, analogous statements hold for copowers and sums.

2. 7. Now, we shall show that also the one cancellation law which is not an immediate consequence of combinatoriality (Theorem 4 in [2]) can be proved in a way analogous to that of Theorem 5 in [3].

Theorem. *Let a, b, c, d be objects of $\mathfrak{A}$ such that there is only a finite number of morphisms between them, let*

$$d \begin{array}{l} \overset{\alpha}{\nearrow} a \\ \underset{\gamma}{\searrow} c \end{array} , \qquad d \begin{array}{l} \overset{\beta}{\nearrow} b \\ \underset{\gamma'}{\searrow} c \end{array}$$

be products. Then there is an isomorphism $\delta: d \to d$ such that $\gamma' \cdot \delta = \gamma$.

Proof. Let A be the full subcategory of $\mathfrak{A}$ generated by a, b, c, d, $Y: A \to \mathrm{Setf}^{A^{op}}$ the Yoneda embedding (see 2. 5), denote by

$$k: Y(d) \to Y(a) \times Y(c), \quad k': Y(d) \to Y(b) \times Y(c)$$

the natural equivalences defined by $k^x(\mu) = (\alpha\mu, \gamma\mu)$, $k'^x(\mu) = (\beta\mu, \gamma'\mu)$. Let B be the category the objects of which are couples (x, ξ) with x an object of A, $\xi: x \to c$, the morphisms from (x, ξ) into (y, η) being the triples (ξ, φ, η) with $\varphi: y \to x$ such that $\xi\varphi = \eta$. $U: B \to A^{op}$ is defined by $U(x, \xi) = x$, $U(\xi, \varphi, \eta) = \varphi$, $U^*: \mathrm{Setf}^{A^{op}} \to \mathrm{Setf}^B$ by $U^*(f) = f \circ U$, $U^*(\tau) = \tau U$. U^* obviously preserves products (see 1. 4. 5)).

Finally, define a functor $L: \mathrm{Setf}^B \to \mathrm{Setf}^{A^{op}}$ by

$$L(f)(x) = U\{f(x, \xi) \times \{\xi\} | \xi: x \to c\}, \quad L(f)(\varphi)(u, \xi) = (f(\xi, \varphi, \xi\varphi)(u), \xi\varphi),$$

and

$$L(\tau)^x(u, \xi) = (\tau^{(x, \xi)}(u), \xi) \quad \text{for} \quad \tau: f \to g.$$

Let $e: B \to \mathrm{Set} f$ be the constant functor defined by $e(\alpha) = 1_{\{0\}}$. It is a singleton of $\mathrm{Set} f^B$ and we have a transformation $\tau: e \to U^* Y(c)$, namely that defined by $\tau^{(x, \xi)}(0) = \xi$.

Thus, since Setf^B is combinatorial and since U^* preserves products, we have by 2. 6. 2) an isomorphism $\vartheta: U^*Y(a) \to U^*Y(b)$. Now, consider $k'^{-1} \cdot L(\vartheta) \cdot k: Y(d) \to \to Y(d)$. Since Y is full, it is equal to some $Y(\delta)$ with $\delta: d \to d$. We have

$$\gamma'\delta = p_2 k'^d(\delta) = p_2\big(L(\vartheta)^d(\alpha, \gamma)\big) = p_2\big(\vartheta^{(d,\gamma)}(\alpha), \gamma\big) = \gamma$$

(p_2 is the natural projection $A(d, b) \times A(d, c) \to A(d, c)$).

§ 3. Embedding of infinite categories into combinatorial ones

3. 1. In this paragraph we shall use the following known statements:

1) Let A, B, C be categories, A small and C cocomplete (or A finite, B quasifinite and C finitely cocomplete). Let $F: A \to B$ be a functor, let $F^*: C^B \to C^A$ be defined by $F^*(f) = f \circ F$, $F^*(\tau) = \tau F$. Then F^* has a left adjoint.

2) If F is a full embedding and L a left adjoint to F^*, then $F^* \circ L \cong 1$.

3) Let C be Set or Setf, $Y_1: A^{op} \to C^A$, $Y_2: B^{op} \to C^B$ the Yoneda embeddings, $F: A \to B$ arbitrary, $F': A^{op} \to B^{op}$ given by the same formula as F, L a left adjoint to F^*. Then $L \circ Y_1 \cong Y_2 \circ F'$.

The first one is, in substance, due to Kan and a proof can be found in [5], p. 74, the second to Freyd. The third is due to Isbell ([1]).

3. 2. Remark. The functor L from 2. 7 is the left adjoint to U^* granted by 3. 1. 1).

3. 3. Theorem. *For every quasifinite category $\mathfrak{A}$ there is a full product preserving embedding into a combinatorial category $\mathfrak{B}$. If $\mathfrak{A}$ is finite, $\mathfrak{B}$ can be found countable, if $\mathfrak{A}$ is small infinite, $\mathfrak{B}$ can be found with $|\mathfrak{A}| = |\mathfrak{B}|$.*

Proof. For a full finite subcategory A of $\mathfrak{A}$ denote by J_A the embedding $A \subset \mathfrak{A}$, by L_A a left adjoint to $J_A^*: \mathrm{Setf}^{\mathfrak{A}^{op}} \to \mathrm{Setf}^{A^{op}}$.

Denote by $\mathfrak{B}$ the full subcategory of $\mathrm{Setf}^{\mathfrak{A}^{op}}$ generated by the functors isomorphic to those of the form $L_A(f)$. By 3. 1. 3), the Yoneda embedding $Y: \mathfrak{A} \to \mathrm{Setf}^{\mathfrak{A}^{op}}$ maps $\mathfrak{A}$ into $\mathfrak{B}$. Thus, a full product preserving embedding is obtained. Now, it suffices to show that $\mathfrak{B}$ is combinatorial. We have $\mathfrak{B}(L_A f, g) \cong \mathrm{Setf}^A(f, J_A^* g)$ always finite. Let $g_i = L_{A_i}(h_i)$, $i = 1, 2$, be such that, for every g, $|\mathfrak{B}(g, g_1)| = |\mathfrak{B}(g, g_2)|$. Denote by A the full subcategory of $\mathfrak{A}$ generated by A_1 and A_2, by J_i the embedding $A_i \subset A$, by L_i a left adjoint to J_i^*. Obviously $L_{A_i} \cong L_A \circ L_i$ and hence $g_i \cong L_A(h_i')$ where $h_i' = L_i(h_i)$. For any $f: A \to \mathrm{Setf}$, by 3. 1. 2),

$$|\mathrm{Setf}^{A^{op}}(f, h_i')| = |\mathrm{Setf}^{A^{op}}(f, J_A^* L_A(h_i'))| = |\mathfrak{B}(L_A f, g_i)|.$$

Thus, since $\mathrm{Setf}^{A^{op}}$ is combinatorial, we have $h_1' \cong h_2'$ and hence $g_1 \cong g_2$.

3.4. Corollary. *Let $\mathfrak{A}$ be a quasifinite category, let S be the partial binary algebra the elements of which are isomorphism types of objects of $\mathfrak{A}$, with product (sum, resp.) as the partial operation. Denote by N the multiplicative semigroup of natural numbers. Then, for M suitably large, S is isomorphic to a subalgebra of N^M.*

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