A note on small categories with zero

By ŠTEFAN SCHWARZ in Bratislava (ČSSR)

Let S be a semigroup with zero 0. We say that an idempotent e is a categorical left unit if ex is either x or 0 for any $x \in S$. An idempotent e is a categorical right unit if xe is either x or 0 for any $x \in S$.

A semigroup with zero is called *categorical at zero* if $abc=0$ implies either $ab=0$ or $bc=0$.

In accordance with [1] we define:

Definition. A semigroup with 0 is called a *small category with zero* if it satisfies the following two conditions:

C_1. To any non-zero element $a \in S$ there is a categorical left unit $e_l(a)$ and a categorical right unit $e_r(a)$ such that $e_l(a) \cdot a = a \cdot e_r(a) = a$.

C_2. S is categorical at zero.

For brevity we shall denote in the following a small category with zero as a C-semigroup.

The connection between category theory and C-semigroups is well known. C-semigroups have been studied by several authors, in particular by HOEHNKE (see e.g. [2]).

The purpose of this note is to prove a theorem concerning the relation between C-semigroups and a class of semigroups called dual semigroups introduced by the author ([3], [4]).

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The following is partly presented in [1] in form of Exercises:

Lemma 1. *In a semigroup satisfying Condition* C_1 *we have:*

a) $e_r(a)$, $e_l(a)$ *are uniquely determined.*
b) $a \in Sa$, $a \in aS$, *in particular* $S^2 = S$.
c) *Any categorical left unit of* S *is a categorical right unit of* S.

With respect to c) we may speak (in a semigroup satisfying C_1) about the set of all non-zero categorical idempotents $\in S$. This set will be denoted by E. E is a subset of the set of all non-zero idempotents E_0 and simple examples show that E can be a proper subset of E_0.

Note that if card $E=1$, i.e. $E=\{e\}$, then e is the two-sided unit element of S.

Let $e_1, e_2 \in E$ and $e_1 \neq e_2$. Then $Se_1 \cap Se_2 = 0$. For if there were $0 \neq a \in Se_1 \cap \cap Se_2$ we would have $a=be_1=ce_2$ $(b, c \in S)$, hence $a=ae_1=ae_2$, a contradiction to a). Since any $a \in S$ can be written in the form $a=ae$, $e \in E$, we have

Lemma 2. *Any semigroup satisfying Condition* C_1 *can be written in the form*

$$S=\bigcup_{e_\alpha \in E} Se_\alpha = \bigcup_{e_\alpha \in E} e_\alpha S,$$

where $Se_\alpha \cap Se_\beta = e_\alpha S \cap e_\beta S = 0$ *for* $e_\alpha \neq e_\beta$.

Lemma 3. *If* $e_1, e_2 \in E$ *and* $e_1 \neq e_2$, *then* $e_1 e_2 = 0$.

Proof. Suppose that $e_1 e_2 \neq 0$. Then since both e_1 and e_2 are categorical units we have $e_1 e_2 = e_1$ and $e_1 e_2 = e_2$, hence $e_1 = e_2$, contrary to the assumption.

We now use Condition C_2.

Lemma 4. *If S is a C-semigroup,* $e \in E$ *and* $0 \neq a \in Se$, $0 \neq b \in eS$, *then* $ab \neq 0$.

Proof. $ab=aeb=0$ implies either $ae=0$ or $eb=0$, i.e. either $a=0$ or $b=0$.

Remark. There are large classes of semigroups satisfying Condition C_2. E.g., a 0-simple semigroup containing a 0-minimal left ideal (in particular a completely 0-simple semigroup) is categorical at zero. (For a proof see [1], Lemma 8, 23, p. 86.)

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In [3] (see also [1], pp. 29—30) we have introduced the notion of a dual semigroup. If S is a semigroup with zero and A a subset of S, the *left* and *right annihilators* $r(A)$ and $l(A)$ are defined by $r(A)=\{x \in S \mid Ax=0\}$, $l(A)=\{x \in S \mid xA=0\}$.

A semigroup with zero is called *dual* if for any left ideal L of S we have $l[r(L)]=L$ and for any right ideal R of S we have $r[l(R)]=R$.

Clearly $l(A)$ and $r(A)$ are left and right ideals, respectively.

It has been proved in [4] that in a dual semigroup S any left ideal of S contains a 0-minimal left ideal of S. If L_0 is a 0-minimal left ideal of S, then $r(L_0)$ is a maximal right ideal of S. Analogously for right ideals. Also if for two left ideals we have $L_1 \subsetneqq L_2$, then $r(L_1) \supsetneqq r(L_2)$.

Recall for further purposes: A semigroup S is called a 0-*direct union* of its two-sided ideals M_α, $\alpha \in A$, if

$$S=\bigcup_{\alpha \in A} M_\alpha \quad \text{and} \quad M_\alpha M_\beta = M_\alpha \cap M_\beta = 0 \quad \text{for} \quad \alpha \neq \beta.$$

We now prove the following

Theorem. *A small category with zero is a dual semigroup if and only if it is a 0-direct union of completely 0-simple dual semigroups.*

Proof. 1. Let S be a C-semigroup. Consider the left ideal Se, $e\in E$. Since $Se\cdot fS=0$ for every $f\neq e$ $(e,f\in E)$, we have $r(Se)\supset\{\bigcup_f fS\,|\,f\in E, f\neq e\}$. By Lemma 4 for any $a\in eS$ we have $Se\cdot a\neq 0$. Hence $r(Se)=\{\bigcup_f fS\,|\,f\in E, f\neq e\}$.

We next prove: If S is moreover dual, then Se is a 0-minimal left ideal of S. Let $L\neq 0$ be a left ideal of S, $L\subsetneqq Se$. By duality we have $r(Se)\subsetneqq r(L)$. Hence $r(L)\supsetneqq \{\bigcup fS\,|\,f\in E, f\neq e\}$ and $r(L)\cap eS\neq 0$. There exists therefore an element $a\in eS$ such that $La=0$. This constitutes a contradiction since for any $b\in L$, $b\neq 0$ and $a\in eS$, $a\neq 0$ we have (by Lemma 4) $ba\neq 0$. Hence Se is a 0-minimal left ideal of S.

Call two ideals $A_1\neq A_2$ of S *quasidisjoint* if $A_1\cap A_2=0$.

Lemma 2 implies that S is a union of pairwise quasidisjoint 0-minimal left ideals of S. Analogously, S is a union of pairwise quasidisjoint 0-minimal right ideals of S. It is known (see e.g. [1], Theorem 6, 39) that such a semigroup is a 0-direct union of completely 0-simple semigroups. Hence we may write

$$S=\bigcup_{\alpha\in A} M_\alpha,\quad M_\alpha M_\beta=M_\alpha\cap M_\beta=0\quad\text{for}\quad \alpha\neq\beta,$$

where M_α $(\alpha\in A)$ runs through all 0-minimal two-sided ideals of S.

Now it is easy to see that a semigroup which is a 0-direct union of its ideals is dual iff each of the ideal components is dual. (For an explicit proof see [4].) Hence each M_α $(\alpha\in A)$ is dual.

2. Suppose conversely that S is a 0-direct union of completely 0-simple dual semigroups: $S=\bigcup_{\alpha\in A} M_\alpha$, $M_\alpha M_\beta=M_\alpha\cap M_\beta=0$ for $\alpha\neq\beta$.

Recall (see [4]) that a completely 0-simple dual semigroup M_α can be written in the form $M_\alpha=\bigcup_{e=e^2\in M_\alpha} M_\alpha e=\bigcup_{e=e^2\in M_\alpha} eM_\alpha$ with $M_\alpha e_1\cap M_\alpha e_2=0$ and $e_1M_\alpha\cap e_2M_\alpha=0$ for $e_1\neq e_2$, and any non-zero idempotent $\in M_\alpha$ is a categorical unit of M_α. This immediately implies that the 0-direct union $S=\bigcup_{\alpha\in A} M_\alpha$ satisfies Condition C_1. Since (by the Remark after Lemma 4 above) each M_α is categorical at zero, S is clearly also categorical at zero. Hence S satisfies Condition C_2. This proves our Theorem.

Remark 1. In [4] we have proved that any dual semigroup satisfies Condition C_1 and, moreover, in any dual semigroup every non-zero idempotent is a categorical unit. This implies:

Corollary 1. *A dual semigroup which is not a 0-direct union of completely 0-simple dual semigroups is not categorical at zero.*

Remark 2. Recall the following notions. An element $a \neq 0$ of a C-semigroup S is called *invertible*, if there is an element $a' \in S$ such that $aa' = e_l(a)$, $a'a = e_r(a)$. Further, a regular semigroup is said to be primitive if each of its non-zero idempotents is primitive. Using the known results mentioned in [1] (p. 79) and the fact proved in [5] (see also [4]) that any 0-simple dual semigroup is completely 0-simple, we have the following

Corollary 2. *For a semigroup with zero the following statements are equivalent:*

1. *S is a small category with zero which is a dual semigroup.*
2. *S is a 0-direct union of 0-simple dual semigroups.*
3. *S is a small category with zero every non-zero element of which is invertible.*
4. *S is a primitive inverse semigroup.*

References

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