# Hausdorff operators

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If $x=\{x_1, x_2, x_3, \dots\}$ is any complex sequence and if $C_0$ is the Cesàro matrix then $(C_0 x)_n$, the $n$th term of the transformed sequence, is given by $\frac{1}{n}\sum_{k=1}^{n} x_k$. The Cesàro matrix belongs to a class of triangular matrices which are known as Hausdorff matrices In [1] A. BROWN, P. R. HALMOS, and A. L. SHIELDS have shown that the Cesàro matrix defines a bounded operator on $l^2$, whose spectrum is $\{z: |z-1| \leqq 1\}$ and that it is hyponormal. Whether or not this operator is subnormal is left by them as an open question. T. L. KRIETE, III, and DAVID TRUTT in [3] show that the Cesàro operator on $l^2$ is subnormal. We show that certain more general Hausdorff matrices as operators on $l^2$ are also subnormal.

In [1] the continuous analogues of the Cesàro matrix are also studied. In particular, it is shown that if $C$ and $C_1$ denote the continuous analogues of $C_0$ on $L^2(0, 1)$ and on $L^2(0, \infty)$, respectively, then $I-C^*$ is a simple unilateral shift and $I-C_1^*$ is a simple bilateral shift. We show that $I-\Gamma_a^{1*}/N$, in the notation of [4], as an operator on $L^2(0, 1)$ (or $L^2(0, \infty)$) is a simple unilateral shift (or a simple bilateral shift) for $a>1/2$. I wish to thank Professors B. E. RHOADES and J. P. WILLIAMS for many fruitful conversations. I am grateful to the referee for his comments which resulted in simplifying the proofs of Theorems 3 and 4.

Theorem 1. *Let $A=(a_{nk})$ be a real triangular matrix which is a bounded operator on $l^2$. Then $\sigma(A)$, the spectrum of $A$, is symmetric about the real axis.*

Proof. Let $\lambda = a+ib$ be any complex number such that $b\neq 0$. Then

$$A-\lambda I = (A-aI)-ibI = b\left[\frac{1}{b}(A-aI)-iI\right].$$

Since $A$ is real, $\frac{1}{b}(A-aI)$ is real. Therefore, it suffices to show that $i$ is not in $\sigma(A)$ if and only $-i$ is not. Since $A-iI$ and $A+iI$ are clearly one-one, it suffices to show that both are simultaneously onto. But if $x$, $y$, $u$, $v$ are real sequences in $l^2$ then

$(A+iI)(x+iy)=u+iv$ implies $(A-iI)(x-iy)=u-iv$ and vice versa. Hence the theorem follows.

Corollary. *Let $H$ be a Hausdorff matrix with real entries which is a bounded operator on $l^2$. Then $\sigma(H)$ is symmetric about the real axis.*

Theorem 2. *Every Hausdorff matrix $H$ that is a bounded operator on $l^2$ is subnormal.*

Proof. We know from [3] that $I-C_0$ is subnormal with a cyclic vector. It follows from [2] that $H$ commutes with $I-C_0$. The theorem now follows from [5].

Definition 1. For each $a>1/2$ let $B_a$ be the operator defined on $L^2(0, \infty)$ by

$$B_a f(x) = f(x)-(2a-1)x^{a-1}\int_x^\infty f(s)s^{-a}\,ds.$$

Note that $B_1 = I-C_1^*$ is the operator studied in [1]. It follows from [4] that $B_a$ is a bounded operator.

Theorem 3. *$B_a$ is a simple bilateral shift on $L^2(0, \infty)$.*

Proof. Consider the map $T_a: L^2(0, \infty)\to L^2(0, \infty)$ such that for any $f$ in $L^2(0, \infty)$

$$T_a f(x) = \sqrt{2a-1}\, x^{a-1} f(x^{2a-1}).$$

A change of variable argument shows that $T_a$ is norm preserving. It is easy to check that $T_a^{-1}=T_b$ where $b=\dfrac{a}{2a-1}$. Hence $T_a$ is unitary. Observe also that for any $f$ in $L^2(0, \infty)$

$$T_a B_1 f = B_a T_a f.$$

Hence $B_a$ is unitarily equivalent to $B_1$. Since $B_1$ is a simple bilateral shift [1, Theorem 5], it follows that $B_a$ is also a simple bilateral shift.

Since the spectrum of a simple bilateral shift is the unit circle, Theorem 3 yields the following result of Rhoades [4, Theorem 16]:

Corollary. *$\sigma(\Gamma_a^{1*}) = \{z\colon |z-N| = N\}$, where $N=\dfrac{2a}{2a-1}$ and $B_a = I-\Gamma_a^{1*}/N$.*

Remark. $\{T_a : a>1/2\}$ forms a unitary group under the composition $T_aT_b = T_{a\oplus b}$ where $a\oplus b = 2ab-a-b+1$ for $a, b>1/2$. Observe that the set of real numbers greater than $1/2$ forms a group under the composition $\oplus$ and that this group is isomorphic to the group of positive real numbers under the map $\alpha\to 2\alpha-1$.

Definition 2. For each $a>1/2$ let $V_a$ be the operator defined on $L^2(0, 1)$ by

$$V_a f(x) = f(x)-(2a-1)x^{a-1}\int_x^1 f(s)s^{-a}\,ds.$$

Note that $V_1 = I-C^*$ is the operator studied in [1].

Theorem 4. *$V_a$ is a simple unilateral shift.*

Proof. Since $L^2(0, 1)$ is invariant under $T_a$, $T_a^{-1}$, $B_a$, and $B_1$, and since, as we have seen in the proof of Theorem 3, $T_a B_1 = B_a T_a$ we get $T_a|_{L^2(0,1)} V_1 = V_a T_a|_{L^2(0,1)}$ because $B_1|_{L^2(0,1)} = V_1$ and $B_a|_{L^2(0,1)} = V_a$. Observe that $T_a|_{L^2(0,1)}$ is also unitary to conclude that $V_a$ is unitarily equivalent to $V_1$. Since $V_1$ is a simple unilateral shift [1, Theorem 4], the theorem follows.

Since the spectrum of a simple unilateral shift is the unit disk, Theorem 4 yields the following result of RHOADES [4, Theorem 10]:

Corollary. *$\sigma(\Gamma_a^{1*}) = \{z : |z-N| \leqq N\}$, where $N = \dfrac{2a}{2a-1}$ and $V_a = I - \Gamma_a^{1*}/_N$.*

Remark. The orthogonal complement of the range of $V_1$ is the one-dimensional subspace spanned by the constant function $e(x) \equiv 1$. Since $V_1$ is a simple unilateral shift it follows that $\{e, V_1 e, V_1^2 e, \dots\}$ is an orthonormal basis for $L^2(0, 1)$. Since

$$V_1^n e(x) = 1 + \binom{n}{1} \log x + \binom{n}{2} \frac{1}{2!} (\log x)^2 + \dots + \frac{1}{n!} (\log x)^n \quad \text{for} \quad n \geqq 1,$$

it follows that *polynomials in* $\log x$ *are dense in* $L^2(0, 1)$. Observe that the map $S: L^2(0, 1) \to L^2(1, \infty)$ given by $Sf(x) = \dfrac{1}{x} f\left(\dfrac{1}{x}\right)$ is unitary. Since $S(\log x)^n = \dfrac{(-1)^n}{x} (\log x)^n$, it follows that *the linear span of* $\left\{\dfrac{1}{x}, \dfrac{\log x}{x}, \dfrac{(\log x)^2}{x}, \dots\right\}$ *is dense in* $L^2(1, \infty)$.

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