Weighted shifts of class $\mathscr{C}_\varrho$

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§ 1. Introduction

In this paper we study weighted shifts of class $\mathscr{C}_\varrho$ and apply the results to obtain some "metric properties" of operators of class $\mathscr{C}_\varrho$. We shall include some known facts for these classes and resume parts of the papers [2] and [3].

We shall consider complex Hilbert spaces only. Operators will be supposed linear and bounded. For the Hilbert space $\mathfrak{H}$ we denote by $\mathscr{L}(\mathfrak{H})$ the algebra of all operators on $\mathfrak{H}$.

Definition. The operator $T\in\mathscr{L}(\mathfrak{H})$ is said to be of class $\mathscr{C}_\varrho$ ($\varrho>0$) if there exist a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ and a unitary operator $U\in\mathscr{L}(\mathfrak{K})$ such that

$$T^n=\varrho P_{\mathfrak{H}}U^n|_{\mathfrak{H}} \qquad (n=1,2,\dots), \tag{1}$$

$P_{\mathfrak{H}}=P$ denoting orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$. The operator U is called the unitary ϱ-dilation of T.

The classes $\mathscr{C}_\varrho$ were introduced by B. Sz.-Nagy and C. Foiaş cf. [1]. Recall the following facts:

a) $\mathscr{C}_\varrho$ is an increasing function of ϱ, i.e. $\mathscr{C}_\varrho\supset\mathscr{C}_\sigma$ for $\varrho>\sigma$.
b) $\mathscr{C}_1$ is the class of contractions (B. Sz.-Nagy).
c) $\mathscr{C}_2$ is the class of numerical radius contractions (C. A. Berger).
d) If $T\in\mathscr{C}_\varrho$, then $\|T^n\|\leqq\varrho$ and $\nu(T)\leqq\min\{1,\varrho\}$ ($\nu(T)$ means spectral radius).
e) $T\in\mathscr{C}_\varrho$ if and only if

$$(\varrho-2)\|zTh\|^2-2(\varrho-1)\operatorname{Re}(zTh,h)+\varrho\|h\|^2\geqq 0 \quad \text{for} \quad h\in\mathfrak{H} \quad \text{and} \quad |z|\leqq 1. \tag{2}$$

We will also use the following obvious corollaries of e):
f) If $T\in\mathscr{C}_\varrho$ and $\mathfrak{H}_0\subset\mathfrak{H}$ is a closed invariant subspace for T, then $T|\mathfrak{H}_0\in\mathscr{C}_\varrho$.
g) The class $\mathscr{C}_\varrho$ is closed in the strong operator topology.

1.1. Proposition. *If $T_j\in\mathscr{L}(\mathfrak{H}_j)$ belongs to the class $\mathscr{C}_{\varrho_j}$ ($j=1,2$), then $T_1\otimes T_2\in\mathscr{C}_{\varrho_1\varrho_2}$.*

Proof. Indeed, if U_j is a unitary ϱ_j-dilation of T_j in $\mathfrak{K}_j \supset \mathfrak{H}_j$, it is easy to verify that $U_1 \otimes U_2$ is a unitary $\varrho_1 \varrho_2$-dilation of $T_1 \otimes T_2$ in $\mathfrak{K}_1 \otimes \mathfrak{K}_2$.

1.2. Proposition. *$T \in \mathscr{C}_\varrho$ if and only if*

(i) $v(T) \leqq a = \min\{1, \varrho\}$,

(ii) $(\varrho-2)\|Th\|^2 - 2|\varrho-1|\,|(Th, h)| + \varrho\|h\|^2 \geqq 0$ *for all* $h \in \mathfrak{H}$.

(i) *is redundant if* $0 < \varrho \leqq 2$.

Proof. The case $\varrho=1$ is obvious. The necessity part follows from d) and e) taking $|z|=1$. Let $\varrho \neq 1$ and suppose that (i) and (ii) are satisfied. Remark that (ii) may be written in the form:

$$(\varrho-2)\|zTh\|^2 - 2(\varrho-1)\,\mathrm{Re}\,(zTh, h) + \varrho\|h\|^2 \geqq 0 \quad \text{for} \quad |z|=1 \quad \text{and} \quad h \in \mathfrak{H}$$

or, equivalently,

$$\|[\varrho I - (\varrho-1)zT]h\| \geqq \|zTh\| \quad \text{for} \quad h \in \mathfrak{H}, \quad |z|=1. \tag{3}$$

From (i) it follows that $\varrho|\varrho-1|^{-1} > a$; hence

$$C(z) = zT[\varrho I - (\varrho-1)zT]^{-1} \in \mathscr{L}(\mathfrak{H}),$$

for $|z| < b$, where $b = \varrho|\varrho-1|^{-1}a^{-1}$. Since $b > 1$, inequality (3) may be written in the form

$$\|C(z)\| \leqq 1 \quad \text{for} \quad |z|=1.$$

$C(z)$ being analytic on the closed unit disk, it follows by the maximum modulus theorem that

$$\|C(z)\| \leqq 1 \quad \text{for} \quad |z| \leqq 1,$$

that is,

$$\|[\varrho I - (\varrho-1)zT]h\| \geqq \|zTh\| \quad \text{for} \quad h \in \mathfrak{H}, \quad |z| \leqq 1,$$

which is equivalent to (2). The proof is complete.

1.3. Recall now a construction from [2]. Let T a power-bounded operator in $\mathscr{L}(\mathfrak{H})$. Put $\boldsymbol{H} = \bigoplus_{-\infty}^{\infty} \mathfrak{H}_k$, where each $\mathfrak{H}_k$ is a copy of $\mathfrak{H}$, and denote by $\{\overset{(k)}{h_k}\}_{k \in \boldsymbol{Z}}$ the elements of $\boldsymbol{H}$. We shall denote an element of the form $\{\dots, 0, \dots, \overset{(k-1)}{0}, \overset{(k)}{h}, \overset{(k+1)}{0}, \dots, 0, \dots\}$ simply by $\overset{(k)}{h}$. Let $\{p_k\}_{k \in \boldsymbol{Z}}$ be an arbitrary sequence of positive integers. Define $\boldsymbol{T} \in \mathscr{L}(\boldsymbol{H})$ as the operator $\overset{(k)}{h} \to \overset{(k+1)}{\overbrace{T^{p_k}h}}$. In [2] it is proved that if $T \in \mathscr{C}_\varrho$, then $\boldsymbol{T} \in \mathscr{C}_\varrho$.

We shall use also the following

1.4. Theorem. *If $T \in \mathscr{C}_\varrho$, the sequence $\{\|T^n h\|\}$ converges for all $h \in \mathfrak{H}$.*

For the proof, see [2] or [7].

§ 2. Weighted bilateral shifts

In this paragraph we shall consider a Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbf{Z}}$ and corresponding *weighted* (bilateral) *shifts*, i.e. operators which transform e_k into $w_k e_{k+1}$, where $\{w_k\}_{k\in\mathbf{Z}}$ is a bounded sequence of complex numbers. Such a weighted shift is unitarily equivalent to the one with weights $\{|w_k|\}$ so we can suppose that the weights are nonnegative (see [5] or [6]).

We shall denote by $\{\ldots, w_{-n}, \ldots, w_{-1}, w_0, w_1, \ldots, w_n, \ldots\}$ or briefly by $\{w_k\}$ the weights as well as the operator itself.

2. 1. Proposition. *If* $\{w_k\}\in\mathscr{C}_\varrho$ *and* $\{s_k\}\in\mathscr{C}_\sigma$ *then* $\{w_k s_k\}\in\mathscr{C}_{\varrho\sigma}$.

Proof. Aplying 1. 1 we have that $\{w_k\}\otimes\{s_k\}\in\mathscr{C}_{\varrho\sigma}$. Notice that the subspace $\mathfrak{H}_0\subset\mathfrak{H}\otimes\mathfrak{H}$ generated by $\{e_k\otimes e_k\}_{k\in\mathbf{Z}}$ is invariant for $\{w_k\}\otimes\{s_k\}$ and the restriction to $\mathfrak{H}_0$ of this operator is also of class $\mathscr{C}_{\varrho\sigma}$. But this restriction is a weighted shift with weights $\{w_k s_k\}$.

2. 2. Corollary. *If* $\{w_k\}\in\mathscr{C}_\varrho$ *and* $0\leqq s_k\leqq w_k$, *then* $\{s_k\}\in\mathscr{C}_\varrho$.

Proof. One can find numbers $0\leqq\alpha_k\leqq 1$ such that $s_k=\alpha_k w_k$. Since $\{\alpha_k\}$ is a contraction, the conclusion follows.

2. 3 Proposition. $T=\{w_k\}\in\mathscr{C}_\varrho$ *if and only if*

(j) $$v(T)\leqq a=\min\{1,\varrho\},$$

(jj) $$\sum_{k=-\infty}^{\infty}[(\varrho-2)w_k^2+\varrho]x_k^2-\sum_{k=-\infty}^{\infty}2(\varrho-1)w_k x_k x_{k+1}\geqq 0$$

for every sequence of real numbers x_k *with* $\sum_{-\infty}^{\infty}x_k^2<\infty$.

Proof. Take $h=\sum z_k e_k$ and apply (1. 2). If we put $x_k=|z_k|$ we obtain (jj).

2. 4. Lemma. *The real infinite quadratic form*

$$\sum_{-\infty}^{\infty}a_k x_k^2-\sum_{-\infty}^{\infty}b_k x_k x_{k+1}\quad\left(\sum_{-\infty}^{\infty}x_k^2<\infty;\ a_k,\ b_k\ \textit{bounded}\right)$$

is positive semidefinite if and only if it can be written in the form

$$\sum(\alpha_k x_k-\beta_k x_{k+1})^2\quad(\alpha_k,\beta_k\in\mathbf{R}).$$

Proof. See [3].

2. 5. Theorem. $\{w_k\}\in\mathscr{C}_2$ *if and only if the weights are of the form* $w_k^2=$ $=(1-c_k)(1+c_{k+1})$, $c_k\in[-1,1]$, $k\in\mathbf{Z}$.

Proof. As known, $\mathscr{C}_2$ consists of the operators T with $|(Th, h)| \leqq \|h\|^2$, that is, of numerical radius contradictions. Using 2. 3, a necessary and sufficient condition for $\{w_k\}$ to be of the class $\mathscr{C}_2$ is that

$$\sum_{-\infty}^{\infty} x_k^2 - \sum_{-\infty}^{\infty} w_k x_k x_{k+1} \geqq 0 \quad \text{if} \quad \sum_{-\infty}^{\infty} x_k^2 < \infty,$$

that is (using 2. 4)

$$\alpha_k^2 + \beta_{k-1}^2 = 1, \; 2\alpha_k \beta_k = w_k, \quad \text{where} \quad \alpha_k, \beta_k \in \mathbf{R}, \; k \in \mathbf{Z}.$$

We have $w_k^2 = 4\alpha_k^2 \beta_k^2 = 2\alpha_k^2(2-2\alpha_{k+1}^2)$. Put $c_k = 1-2\alpha_k^2$, and the conclusion follows.

2. 6. Proposition. *$T = \{w_k\} \in \mathscr{C}_\varrho$ $(\varrho > 2)$ if and only if*

$$\text{(k)} \quad \nu(T) \leqq 1, \qquad \text{(kk)} \quad \{u_k\} \in \mathscr{C}_2,$$

where

$$u_k = \frac{2(\varrho-1)w_k}{\sqrt{(\varrho-2)w_k^2+\varrho}\,\sqrt{(\varrho-2)w_{k+1}^2+\varrho}}.$$

Proof. Take $y_k^2 = [(\varrho-2)w_k^2 + \varrho]x_k^2$ in (jj) of (2. 3).

2.7 Proposition. *If $T = \{w_k\} \in \mathscr{C}_\varrho$ then $\prod_{-\infty}^{\infty} w_k$ converges (possibly to 0).*

Observe that $\prod_{-\infty}^{\infty} w_k = \lim \|T^n e_0\| \cdot \lim \|T^{*n} e_0\|$; the limits i on the right hand side exist by 1. 4.

Observe that $\prod_{-\infty}^{\infty} w_k \neq 0$ implies $w_k \to 1$ as $k \to \pm\infty$.

2. 8. Proposition. *If $\{w_k\} \in \mathscr{C}_2$ then $\prod_{-\infty}^{\infty} w_k \leqq 1$.*

Indeed, from 2. 5 we have

$$\prod w_k = \prod \sqrt{(1-c_k)(1+c_{k+1})} = \prod \sqrt{1-c_k^2} \leqq 1.$$

2. 9. Definition. Let $\{w_k\}$ be a weighted shift. A *compression* of $\{w_k\}$ is any weighted shift obtained by substituting a finite sequence of consecutive weights by their product.

For example, $\{\dots, w_{-2}, w_{-1}w_0, w_1, \dots\}$ and $\{\dots, w_{-2}, w_{-1}w_0w_1, w_2, \dots\}$ are compressions of the shift $\{w_k\}$.

2. 10. Proposition. *Every compression of a weighted shift $\{w_k\}$ of class $\mathscr{C}_\varrho$ is also of class $\mathscr{C}_\varrho$.*

Proof. Choose $m \leqq n$ and let $\{v_k\}$ be the weighted shift with

$$v_k = w_k \quad \text{for} \quad k < m, \qquad v_m = w_m \dots w_n, \qquad v_k = w_{k+n-m} \quad \text{for} \quad k > m.$$

To prove that $\{v_k\} \in \mathscr{C}_\varrho$ we shall repeat the construction of 1. 3 by choosing $p_k = 1$ for $k \neq m$ and $p_m = n-m+1$. Let $\mathfrak{H}_0$ be the subspace of $\boldsymbol{H}$ with base $\{\overset{(k)}{e_k}\}$ for $k \leqq m$, and $\{e^{(k)}_{k+n-m}\}$ for $k > m$. $\mathfrak{H}_0$ will be invariant for $\boldsymbol{T}$ and $\boldsymbol{T}|\mathfrak{H}_0$ will be just the weighted shift with weights $\{v_k\}$.

2. 11 Proposition. *If $\{w_k\} \in \mathscr{C}_\varrho$ then $a = \Pi w_k \leqq 1$.*

Proof. For $\varrho = 2$, (and then also for $\varrho < 2$) this is contained in 2. 8. Denote by T_n the weighted shift obtained from $T = \{w_k\}$ by compression of weights from w_{-n} to w_n. By 2. 10, $T_n \in \mathscr{C}_\varrho$. If $a > 0$, then $T_n \to \{\dots, 1, \dots, 1, a, 1, \dots, 1, \dots\}$ (strongly). It follows that $\{\dots, 1, a, 1, \dots\} \in \mathscr{C}_\varrho$. If $a > 1$, by Corollary 2. 2 we may suppose $1 < a < \dfrac{\varrho}{\varrho - 2}$. Using 2. 6 we deduce that

$$u_k = \left\{\dots, 1, \dots, 1, \sqrt{\frac{2(\varrho-1)}{(\varrho-2)a^2+\varrho}}, \sqrt{\frac{2(\varrho-1)}{(\varrho-2)a^2+\varrho}} \cdot a, 1, \dots\right\} \in \mathscr{C}_2.$$

But $1 \geqq \prod u_k = \dfrac{2(\varrho-1)a}{(\varrho-2)a^2+\varrho} > 1$ $\left(\text{since } a < \dfrac{\varrho}{\varrho-2}\right)$ which is impossible.

2. 12. Theorem. *If $\{w_k\} \in \mathscr{C}_\varrho$ and $\prod\limits_{-\infty}^{\infty} w_k = 1$, then $w_k = 1$ for every $k \in \boldsymbol{Z}$.*

Proof. We may suppose $\varrho > 2$. Suppose some w_k differ from 1. Then we find an m such that $\prod\limits_{-\infty}^{m} w_k = a \neq 1$. Compressing weights from w_{m-n} to w_m and taking $n \to \infty$ it follows that $\{\dots, 1, a, w_{m+1}, \dots\} \in \mathscr{C}_\varrho$. Compressing weights from w_{m+1} to w_{m+n} and passing to limit, we deduce $\{\dots, 1, a, a^{-1}, 1, \dots\} \in \mathscr{C}_\varrho$. Considering, if necessary, the adjoint shift we may assume that $a < 1$. Now using 2. 6 we obtain:

$$u_k = \left\{\dots, 1, \dots 1, \sqrt{\frac{2(\varrho-1)}{(\varrho-2)a^2+\varrho}}, \frac{2(\varrho-1)a^2}{\sqrt{(\varrho-2)a^2+\varrho}\,\sqrt{(\varrho-2)+\varrho a^2}}, \right.$$

$$\left. \sqrt{\frac{2(\varrho-1)}{(\varrho-2)+a^2}}, 1, \dots\right\} \in \mathscr{C}_2.$$

Using 2. 5 we deduce

$$u_k = (1-c_k)(1+c_{k+1}) = \begin{cases} 1 & \text{for} \quad |k|>1, \\ \dfrac{2}{a^2+1-\varepsilon} & \text{for} \quad k=-1, \\ \dfrac{4a^2}{(a^2+1)^2-\varepsilon^2} & \text{for} \quad k=0, \\ \dfrac{2}{a^2+1+\varepsilon} & \text{for} \quad k=1, \end{cases}$$

where we have put $\varepsilon = \dfrac{a^2-1}{\varrho-1}$. By the fact that $\mathscr{C}_\varrho$ is an increasing function of ϱ we may suppose $|\varepsilon|<1$. We have

$$1 = \prod_{-\infty}^{-2} u_k = (1+c_{-1}) \prod_{-\infty}^{-2} (1-c_k^2); \quad \text{hence} \quad c_{-1} \geqq 0.$$

By the same method, from $\prod\limits_{2}^{\infty} u_k = 1$ it follows that $c_2 \leqq 0$. Then,

$$(1-c_{-1})(1+c_0) = \frac{2}{1+a^2-\varepsilon}, \quad (1-c_0)(1+c_1) = \frac{4a^2}{(1+a^2)^2-\varepsilon^2},$$

and

$$(1-c_1)(1+c_2) = \frac{2}{1+a^2+\varepsilon}.$$

From the first equality and from $c_{-1} \geqq 0$ we deduce

$$1-c_0 \leqq \frac{2a^2-2\varepsilon}{1+a^2-\varepsilon},$$

while from the last one and from $c_2 \leqq 0$ we have

$$1+c_1 \leqq \frac{2a^2+2\varepsilon}{1+a^2+\varepsilon}.$$

Hence,

$$\frac{2a^2-2\varepsilon}{1+a^2-\varepsilon} \cdot \frac{2a^2+2\varepsilon}{1+a^2+\varepsilon} \geqq (1-c_0)(1+c_1) = \frac{4a^2}{(1+a^2)^2-\varepsilon^2}$$

and it follows that $\varepsilon = \dfrac{a^2-1}{\varrho-1} = 0$, $a=1$, a contradiction. The proof is complete.

2. 13. Corollary. *If $T=\{w_k\}$ is invertible and $T \in \mathscr{C}_\varrho$, $T^{-1} \in \mathscr{C}_\varrho$, then T is unitary.*

Proof. It suffices to remark that T^{-1} is also a weighted shift with weights $\{w_k^{-1}\}$. Using 2. 11 we deduce $\prod w_k \leqq 1$ and $\prod (w_k^{-1}) \leqq 1$ hence $\prod w_k = 1$, that is (from 2. 12) $w_k=1$ for every $k \in \mathbf{Z}$.

§ 3. Invertible operators of class $\mathscr{C}_\varrho$

Let $\mathfrak{H}$ be a Hilbert space and T an invertible operator of class $\mathscr{C}_\varrho$.

3. 1. Theorem. *If* $0 \neq h \in \mathfrak{H}$ *and* $w_k = \dfrac{\|T^{k+1}h\|}{\|T^k h\|}$ $(k \in \boldsymbol{Z})$ *then* $\{w_k\}$ *is a weighted shift of class* $\mathscr{C}_\varrho$.

Proof. We construct, as in 1. 3, the space $\boldsymbol{H}$ and the operator $\boldsymbol{T}$ with all $p_i = 1$. Put $\boldsymbol{h}_k = \overbrace{T^k h}^{(k)}$ $(k \in \boldsymbol{Z})$. Let $\mathfrak{H}_0$ be the subspace $\bigvee_{k=-\infty}^{\infty} \boldsymbol{h}_k$. Then $\mathfrak{H}_0$ has the orthonormal basis $e_k = \dfrac{\boldsymbol{h}_k}{\|\boldsymbol{h}_k\|}$.

It is easy to see that $\boldsymbol{T}$ leaves $\mathfrak{H}_0$ invariant, and $\boldsymbol{T}|_{\mathfrak{H}_0}$ is just the desired weighted shift. Using 1. 7 and 1. 2 the proof is complete.

3. 2. Corollary. *If* $T \in \mathscr{C}_\varrho$ *and* T *is invertible, then*

$$\lim \|T^n h\| \leqq \lim \|T^{-n} h\| \quad \textit{for} \quad h \in \mathfrak{H}.$$

Proof. Using 2. 11 and 3. 1 we have

$$1 \geqq \prod \|T^{k+1}h\| \cdot \|T^k h\|^{-1} = \frac{\lim \|T^n h\|}{\lim \|T^{-n} h\|}.$$

3. 3. Corollary. *If* $T \in \mathscr{C}_\varrho$, *$T$ is invertible, and* $\lim \|T^n h\| = \lim \|T^{-n} h\|$, *then*

$$\|T^n h\| = \|h\| \quad \textit{for} \quad n = 1, 2, \ldots.$$

Proof. Obvious from 2. 12 and 3. 1.

3. 4. Corollary. *If* $T \in \mathscr{C}_\varrho$ *and* $\lim \|T^n h\| = \lim \|T^{-n} h\|$ *for all* $h \in \mathfrak{H}$, *then T is unitary.*

3. 5. Corollary. (Stampfli [4].) *If* T, T^{-1} *are both of class* $\mathscr{C}_\varrho$, *then T is unitary.*

Proof. Obvious from 3. 2 and 3. 4.

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