On Σ-ordered inverse semigroups

By S. SRIBALA in Madras (India)

In this paper we define Σ-ordered inverse semigroups (Definition 1) which are partially ordered inverse semigroups whose partial orders are completely determined by families of normal sub-semigroups (Theorem 2). The set of normal sub-semigroups determining the partial order is analogous to the positive cone in a partially ordered group ([2]). We next consider the set $\mathfrak{D}(X)$ of all partial o-isomorphism between o-subsets (Definition 6) of a partially ordered set $(X, \leqq)$. This set is an inverse sub-semigroup of the symmetric inverse semigroup $\mathfrak{I}(X)$ on X. A partial order is defined on $\mathfrak{D}(X)$ in a natural way which makes $\mathfrak{D}(X)$ a Σ-ordered inverse semigroup (Theorem 11). We call $\mathfrak{D}(X)$ the symmetric Σ-ordered inverse semigroup on $(X, \leqq)$. Finally, we prove the Preston—Vagner theorem for Σ-ordered inverse semigroups claiming that any Σ-ordered inverse semigroup can be embedded o-isomorphically into a symmetric Σ-ordered inverse semigroup of partial o-isomorphisms (Theorem 12). Questions of order theoretic interest will be studied separately and are not dealt with in this paper.

For terminology and information on semigroups we refer to [1].

Let S be an inverse semigroup and E the set of idempotents of S. Let Σ be the lattice of all idempotent separating congruence on S with greatest element μ. If M_e $(e\in E)$ is the μ-class containing e, it is known that M_e is a normal subgroup of H_e.

Definition 1. A partially ordered inverse semigroup $(S, \leqq)$ is called a *Σ-ordered inverse semigroup* (and $\leqq$ a Σ-order on S) if the following conditions hold in S:

(1) $a\leqq b \Rightarrow a\mu b$.

(2) If $Q_e=\{x\in S \mid e\leqq x\}$, then $a^{-1}Q_e a\subseteqq Q_{a^{-1}ea}$ for all $e\in E$ and $a\in S$.

Note that in a Σ-ordered semigroup no two idempotents are comparable.

Theorem 2. *Let S be an inverse semigroup and $\{Q_e : e\in E\}$ a collection of subsets which satisfies the following conditions:*

(i) *Q_e is a sub-semigroup of M_e containing e and $Q_e\cdot Q_f \subseteqq Q_{ef}$.*

(ii) *$a^{-1}Q_e a\subseteqq Q_{a^{-1}ea}$ for all $e\in E$ and $a\in S$.*

(iii) *$Q_e\cap Q_e^{-1} = \{e\}$, where Q_e^{-1} is the set of inverses of elements of Q_e in H_e. Then there exists a partial order $\leqq$ on S such that $Q_e=\{x\in S \mid e\leqq x\}$ and $(S, \leqq)$ is a Σ-ordered inverse semigroup.*

Proof. If a partial order $\leqq$ exists on S such that $(S, \leqq)$ is a partially ordered inverse semigroup, then it is clear that $(S, \leqq)$ is a Σ-ordered inverse semigroups. Hence it is enough to establish the existence of such a partial order on S. Let $a, b \in S$. Say $a \leqq b \Leftrightarrow a\mu b,\ ba^{-1} \in Q_{aa^{-1}},\ a^{-1}b \in Q_{a^{-1}a}$. Clearly $a \leqq a$. Let $a, b \in S$, $a \leqq b$ and $b \leqq a$. Then $a \leqq b \Rightarrow a\mu b,\ ba^{-1} \in Q_{aa^{-1}},\ a^{-1}b \in Q_{a^{-1}a}$; $b \leqq a \Rightarrow b\mu a,\ ab^{-1} \in Q_{bb^{-1}},\ b^{-1}a \in Q_{b^{-1}b}$. Now, $a\mu b \Rightarrow aa^{-1}=bb^{-1}=e$ and $a^{-1}a=b^{-1}b=f$. ba^{-1} and ab^{-1} are inverses of each other in Q_e and $a^{-1}b$ and $b^{-1}a$ are inverses of each other in Q_f and so $a=b$. Thus $\leqq$ is asymmetric. To show transitivity, let $a \leqq b$, $b \leqq c$,

$$a \leqq b \Rightarrow a\mu b, \quad ba^{-1} \in Q_{aa^{-1}}, \quad a^{-1}b \in Q_{a^{-1}a}.$$

$$b \leqq c \Rightarrow b\mu c, \quad cb^{-1} \in Q_{bb^{-1}}, \quad b^{-1}c \in Q_{b^{-1}b}$$

Thus $a\mu b\mu c$ and $aa^{-1}=bb^{-1}=cc^{-1}=e$ and $a^{-1}a=b^{-1}b=c^{-1}c=f$ so, $ca^{-1} = cb^{-1}ba^{-1} \in Q_e \cdot Q_e \subseteqq Q_e$ and $a^{-1}c = a^{-1}bb^{-1}c \in Q_f \cdot Q_f \subset Q_f$ and so $a \leqq c$. Thus $\leqq$ is a partial order on S.

If $x \in S$ and $e \leqq x$, then $x\mu e$ and $x=xe=ex \in Q_e$ and so $\{x \in S \mid e \leqq x\} \subseteqq Q_e$. On the otherhand if $x \in Q_e$, then $x\mu e$ and $xe=ex=x \in Q_e$ and so $e \leqq x$ and $Q_e \subseteqq \{x \in S \mid e \leqq x\}$. Thus we have $Q_e = \{x \in S \mid e \leqq x\}$.

Let $a \leqq b$, $c \in S$. Then $a \leqq b \Rightarrow a\mu b,\ ba^{-1} \in Q_{aa^{-1}},\ a^{-1}b \in Q_{a^{-1}a}$; also $a\mu b \Rightarrow ac\mu bc$: $(bc)(ac)^{-1} = bcc^{-1}a^{-1} = ba^{-1}acc^{-1}a^{-1} \in Q_{aa^{-1}} \cdot Q_{(ac)(ac)^{-1}} \subseteqq Q_{(ac)(ac)^{-1}}$; $(ac)^{-1}(bc) = c^{-1}a^{-1}bc \in c^{-1}Q_{a^{-1}a} \cdot c \subseteqq Q_{(ac)^{-1}(ac)}$. Thus $ac \leqq bc$. Similarly we can show that $ca \leqq cb$. Hence $(S, \leqq)$ is a partially ordered inverse semigroup. It now follows that $(S, \leqq)$ is a Σ-ordered inverse semigroup.

Let $(S, \leqq)$ be a Σ-ordered inverse semigroup. Then,

Lemma 3. $a \leqq b \Rightarrow b^{-1} \leqq a^{-1}$.

Proof. $a \leqq b \Rightarrow a\mu b \Rightarrow a^{-1}\mu|b^{-1}|$. Hence $b^{-1} = b^{-1}aa^{-1} \leqq b^{-1}ba^{-1} = a^{-1}$.

Proposition 4. *If T is an inverse sub-semigroup of $(S, \leqq)$, then $(T, \leqq)$ is a Σ-ordered inverse semigroup.*

Proof. Let μ_0 be the maximum idempotent separating congruence on T and μ_T the restriction of μ to T. Then $\mu_T \subseteqq \mu_0$. Let $Q_e^T = Q_e \cap T$. Now $Q_e^T \neq \emptyset \Leftrightarrow e \in T$. The nonempty sets Q_e^T $(e \in E \cap T)$ satisfy the conditions of Theorem 2 and so there exists a partial order $\leqq'$ on T such that $(T, \leqq')$ is a Σ-ordered inverse semigroup. We now show that $\leqq$ coincides with $\leqq'$ on T. It is immediate that $a \leqq b$ $(a, b \in T) \Rightarrow a \leqq' b$. Conversely if $a, b \in T$ then

$$a \leqq' b \Rightarrow \{a\mu_0 b,\ ba^{-1} \in Q^T_{aa^{-1}},\ a^{-1}b \in Q^T_{a^{-1}a}\} \Rightarrow$$
$$\Rightarrow \{aa^{-1}=bb^{-1}=e,\ a^{-1}a=b^{-1}b=f,\ ba^{-1} \in Q_e \subset M_e,\ a^{-1}b \in Q_f \subseteqq M_f\} \Rightarrow$$
$$\Rightarrow \{a\mu b,\ a^{-1}b \in Q_f,\ ba^{-1} \in Q_e\} \Rightarrow a \leqq b.$$

The following lemma follows immediately.

Lemma 5. *$(H_e, \leqq)$, where H_e is the maximal subgroup containing e in $(S, \leqq)$, is a partially ordered group. Further, if $e\mathcal{D}f$, then H_e and H_f are o-isomorphic.*

We shall now describe the symmetric Σ-ordered inverse semigroup on a partially ordered set. Let $(X, \leqq)$ be a partially ordered set. Let $\mathfrak{J}(X)$ denote the symmetric inverse semigroup of all partial $(1-1)$ transformations on X. If $\alpha \in \mathfrak{J}(X)$ denote by $\Delta(\alpha)$ and $\nabla(\alpha)$ the domain and the range of α, respectively.

Definition 6. A subset $A \subseteqq (X, \leqq)$ is called an *o-subset* of X if $a \in A$, $a \leqq b$ (or $b \leqq a$) implies $b \in A$.

It is clear that the set of all o-subsets of $(X, \leqq)$ is closed for the operations of set-intersection and set union and contains the null set φ and X.

Let $\mathfrak{O}(X)$ denote the subset of $\mathfrak{J}(X)$ consisting of all o-isomorphisms between o-subsets of X. $\mathfrak{O}(X)$ contains the map 0 and the identity map of X.

Proposition 7. *$\mathfrak{O}(X)$ is an inverse sub-semigroup of $\mathfrak{J}(X)$.*

Proof. If $\alpha \in \mathfrak{O}(X)$, then α^{-1} is also an o-isomorphism between o-subsets and so belongs to $\mathfrak{O}(X)$. Thus it is enough to show that $\mathfrak{O}(X)$ is a subsemigroup of $\mathfrak{J}(X)$. Let $\alpha, \beta \in \mathfrak{O}(X)$ and $A = \nabla(\alpha) \cap \Delta(\beta)$. If $A = \emptyset$, then $\alpha\beta = 0 \in \mathfrak{O}(X)$. If $A \neq \emptyset$, let $A_1 = A\alpha^{-1}$, $A_2 = A\beta$. A is an o-subset of X. A_1, and A_2 are also o-subsets. For, let $x \in A_1$, $y \leqq x$. Since $A_1 \subset \Delta(\alpha)$, $y \in \Delta(\alpha)$ and so $y\alpha \leqq x\alpha \in A$. Since A is an o-subset, $y\alpha \in A$ and so $y \in A\alpha^{-1} = A_1$. Now $A_1 = \Delta(\alpha\beta)$, $A_2 = \nabla(\alpha\beta)$ and $\alpha\beta : A_1 \to A_2$ is clearly an o-isomorphism with $\beta^{-1}\alpha^{-1}$ as its inverse. Thus $\mathfrak{O}(X)$ is an inverse sub-semigroup of $\mathfrak{J}(X)$.

Definition 8. For $\alpha, \beta \in \mathfrak{O}(X)$ put $\alpha \leqq \beta \Leftrightarrow \alpha \mathscr{H} \beta$, $x\alpha \leqq x\beta$ for all $x \in \Delta(\alpha)$ and $y\beta^{-1} \leqq y\alpha^{-1}$ for all $y \in \nabla(\alpha)$.

Note that $\alpha \mathscr{H} \beta \Leftrightarrow \Delta(\alpha) = \Delta(\beta)$ and $\nabla(\alpha) = \nabla(\beta)$.

Proposition 9. $\alpha \leqq \beta \Rightarrow \alpha\mu\beta$, *where μ is the maximum idempotent separating congruence on $\mathfrak{O}(X)$.*

Proof. We will show that $\alpha \leqq \beta$ implies $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta$ for all idempotents ε in $\mathfrak{O}(X)$ which proves that $\alpha\mu\beta$ ([1] Lemma 7. 57). $\alpha \leqq \beta \Rightarrow \alpha\mathscr{H}\beta$, $x\alpha \leqq x\beta$ for all $x \in \Delta(\alpha)$ and $y\beta^{-1} \leqq y\alpha^{-1}$ for all $y \in \nabla(\alpha)$. Let $A : \Delta(\varepsilon) \cap \Delta(\alpha) = \Delta(\varepsilon) \cap \Delta(\beta)$ and $B = A\varepsilon \cap \Delta(\alpha) = A \cap \Delta(\alpha)$, since ε is identity on A. Then ε is also identity on B and so $\alpha^{-1}\varepsilon\alpha$ is the identity map of $B\alpha$ and $\beta^{-1}\varepsilon\beta$ is the identity map of $B\beta$. Since $\alpha, \beta, \varepsilon$ are o-isomorphism and $\alpha \leqq \beta$ we have A and B are o-subsets and so $B\alpha = B\beta$. Hence $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta$.

Proposition 10. *Let $Q_\varepsilon = \{\alpha \in \mathfrak{O}(X) | \varepsilon \leqq \alpha\}$, where ε is an idempotent of $\mathfrak{O}(X)$. Then $Q_\varepsilon \cdot Q_{\varepsilon_1} \subseteqq Q_{\varepsilon\varepsilon_1}$, $v^{-1} Q_\varepsilon v \subseteqq Q_{v^{-1}\varepsilon v}$ for all elements v and idempotents $\varepsilon, \varepsilon_1 \in \mathfrak{O}(X)$.*

Proof. Let $\alpha \in Q_\varepsilon$, $\beta \in Q_{\varepsilon_1}$. Then $\varepsilon\mu\alpha$ and $\varepsilon_1\mu\beta$ and so $\varepsilon\varepsilon_1\mu\alpha\beta$. Now $\Delta(\varepsilon) \cap \Delta(\varepsilon_1) = \Delta(\varepsilon\varepsilon_1) = \Delta(\alpha\beta) = \nabla(\alpha\beta)$. $x \in \Delta(\varepsilon\varepsilon_1) \Rightarrow x \leqq x\alpha$ and $x \leqq x\beta \Rightarrow x \leqq x\beta \leqq x\alpha\beta$. $y \in \Delta(\varepsilon\varepsilon_1) \Rightarrow y\alpha^{-1} \leqq y$ and $y\beta^{-1} \leqq y \Rightarrow y\beta^{-1}\alpha^{-1} \leqq y$. Thus $\varepsilon\varepsilon_1\mu\alpha\beta$, $x\varepsilon\varepsilon_1 \leqq x\alpha\beta$ and $x\beta^{-1}\alpha^{-1} \leqq x\varepsilon\varepsilon_1$ for all $x \in \Delta(\varepsilon\varepsilon_1)$ and hence $\varepsilon\varepsilon_1 \leqq \alpha\beta$.

Let $\alpha \in Q_\varepsilon$ and $\nu \in \mathfrak{D}(X)$. Then $\nu^{-1}\varepsilon\nu\mu\nu^{-1}\alpha\nu$. So, $\Delta(\nu^{-1}\varepsilon\nu)=\Delta(\nu^{-1}\alpha\nu)=\nabla(\nu^{-1}\alpha\nu)$. $x\in\Delta(\nu^{-1}\varepsilon\nu)$ and $\varepsilon \leqq \alpha \Rightarrow x\nu^{-1}\in\Delta(\varepsilon)\cap\Delta(\nu) = \Delta(\alpha)\cap\Delta(\nu) \Rightarrow x\nu^{-1}\varepsilon \leqq x\nu^{-1}\alpha$ in $\Delta(\nu) \Rightarrow x\nu^{-1}\varepsilon\nu \leqq x\nu^{-1}\alpha\nu$. $y\in\Delta(\nu^{-1}\varepsilon\nu) \Rightarrow y\nu^{-1}\alpha^{-1}\leqq y\nu^{-1}\varepsilon \Rightarrow y\nu^{-1}\alpha^{-1}\nu\leqq y\nu^{-1}\varepsilon\nu$. Thus $\nu^{-1}\varepsilon\nu\leqq\nu^{-1}\alpha\nu$.

Theorem 11. $(\mathfrak{D}(X), \leqq)$ *is a Σ-ordered inverse semigroup. It is called the symmetric Σ-ordered inverse semigroup on* $(X, \leqq)$.

Proof. From propositions 9 and 10 it follows that the sets $Q_\varepsilon=\{\alpha\in\mathfrak{D}(X)|\varepsilon\leqq\alpha\}$ satisfy the conditions (i) and (ii) of Theorem 2. To show condition (iii) consider $\alpha, \alpha^{-1}\in Q_\varepsilon$. $x\in\Delta(\varepsilon)$, $\varepsilon\leqq\alpha \Rightarrow x=x\varepsilon\leqq x\alpha \Rightarrow x\varepsilon\alpha^{-1}\leqq x\alpha\alpha^{-1}=x\varepsilon$; and $\varepsilon\leqq\alpha^{-1} \Rightarrow x= =x\varepsilon\leqq x\alpha^{-1} \Rightarrow x\varepsilon\alpha\leqq x\alpha^{-1}\alpha=x\varepsilon$. Thus we have $x\varepsilon\leqq x\alpha\leqq x\varepsilon$ and $x\varepsilon\leqq x\alpha^{-1}\leqq x\varepsilon$ for all $x\in\Delta(\varepsilon)$ and so $\alpha=\varepsilon=\alpha^{-1}$. It then follows that $\leqq$ is a Σ-order on $\mathfrak{D}(X)$ defined by the sets $\{Q_\varepsilon\}$. Hence $(\mathfrak{D}(X), \leqq)$ is a Σ-ordered inverse semigroup.

We now consider representation of Σ-ordered inverse semigroups by partial transformation.

Proposition 12. *If $(S, \leqq)$ is a Σ-ordered inverse semigroup, then the mapping $\varrho_a\colon x\to xa$ of Sa^{-1} onto Sa, belongs to $\mathfrak{D}(S)$ for every $a\in S$.*

Proof. The sets Sa $(a\in S)$ are o-subsets of $(S, \leqq)$. For, $x\in Sa$, $y\leqq x$ (or $x\leqq y$) $\Rightarrow$ $\Rightarrow x\mu y \Rightarrow Sy=Sx\subseteqq Sa \Rightarrow y\in Sa$. Further ϱ_a and $\varrho_{a^{-1}}$ are order preserving maps which are inverse of each other. Hence $\varrho_a\in\mathfrak{D}(S)$:

Theorem 12. *A Σ-ordered inverse semigroup $(S, \leqq)$ is o-isomorphic to a Σ-ordered inverse sub-semigroup of the symmetric Σ-ordered inverse semigroup $(\mathfrak{D}(S), \leqq)$.*

Proof. The mapping $\varrho\colon a\to\varrho_a$ of S into $\mathfrak{D}(S)$ is clearly an isomorphism of the inverse semigroups. We now show that for $a, b\in S$, $a\leqq b \Leftrightarrow \varrho_a\leqq\varrho_b\cdot a\leqq b\Rightarrow$ $\Rightarrow a\mu b \Rightarrow Sa=Sb$, $Sa^{-1}=Sb^{-1} \Rightarrow \varrho_a\mathscr{H}\varrho_b$. Further, if $x\in Sa^{-1}$, then $x\varrho_a=xa\leqq$ $\leqq xb=x\varrho_b$ and if $y\in Sa$, $y\varrho_b^{-1}=y\varrho_{b^{-1}}=yb^{-1}\leqq ya^{-1}\Rightarrow y\varrho_{a^{-1}}=y\varrho_a^{-1}$. Hence $\varrho_a\leqq\varrho_b$. Next, if $\varrho_a\leqq\varrho_b$, then $Sa^{-1}=\Delta(\varrho_a)=\Delta(\varrho_b)=Sb^{-1}$ and $Sa=\nabla(\varrho_a)=\nabla(\varrho_b)=Sb$, and for all $x\in Sa$, $xa\leqq xb$. Hence $aa^{-1}=bb^{-1}=e$ and $a^{-1}a=b^{-1}b=f$ and so $ea\leqq$ $\leqq eb\cdot i\cdot ea\leqq b$. Hence ϱ is an o-isomorphism of $(S, \leqq)$ into $(\mathfrak{D}(S), \leqq)$.

References

[1] A. H. CLIFFORD-G. B. PRESTON, *Algebraic Theory of Semigroups,* Vol. I. & II, Amer. Math. Soc. Survey 7 (1961, 1967).

[2] L. FUCHS, *Partially ordered algebraic systems,* Pergamon Press (1963).

THE RAMANUJAN INSTITUTE FOR
UNIVERSITY OF MADRAS
MADRAS-5. INDIA

(Received March 30, 1971)