Thin operators in a von Neumann algebra

By CATHERINE L. OLSEN in Toronto (Canada)

1. Introduction. Let $\mathscr{A}$ be a von Neumann algebra, $\mathscr{I}$ a uniformly closed two-sided ideal in $\mathscr{A}$ and $\mathscr{P}$ the lattice of projections in $\mathscr{I}$. Let the center, $\mathscr{Z}$, of $\mathscr{A}$ be identified with $C(\Omega)$, the algebra of all continuous complex-valued functions on some Hyperstonian space [3]. We say that $A \in \mathscr{A}$ is *thin* relative to $\mathscr{I}$ if $A = Z+K$, $Z \in \mathscr{Z}$, $K \in \mathscr{I}$. It is shown that the thin operators relative to $\mathscr{I}$ form a C^*-subalgebra of $\mathscr{A}$. The lattice $\mathscr{P}$ is a directed set under the usual ordering (if $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$, then $P \leqq Q$ means $(Px, x) \leqq (Qx, x)$ all $x \in \mathscr{H}$). It was conjectured by P. R. HALMOS, and proved by R. G. DOUGLAS and C. PEARCY [5] for $\mathscr{A} = \mathscr{B}(\mathscr{H})$, $\mathscr{H}$ separable, and $\mathscr{I}$ the ideal of compact operators, that A is thin relative to $\mathscr{I}$ if and only if

$$(H) \qquad \lim_{P \in \mathscr{P}} \|PAP - AP\| = 0.$$

Douglas and Pearcy asked whether (H) characterizes the thin operators relative to an arbitrary uniformly closed ideal in any von Neumann algebra. It is the purpose of this note to show that this characterization holds for any such ideal in a von Neumann factor. Also, it is proved for maximal ideals in certain more general von Neumann algebras, and for certain ideals in type I algebras.

2. Let $\mathscr{I}$ be a uniformly closed ideal in a von Neumann algebra $\mathscr{A}$. The set of thin operators $\mathscr{I} + \mathscr{Z}$ forms a C^*-subalgebra of $\mathscr{A}$ [4, 1. 8. 4]. There is a two-sided (not necessarily closed) ideal $\mathscr{L}$ of $\mathscr{A}$ with the property that $T \in \mathscr{L}$ if and only if the range projection of T is also in $\mathscr{L}$; furthermore, $\mathscr{I}$ is the uniform closure of $\mathscr{L}$. This fact, due to W. WILS [10, p. 56, Theorem 1. 4] will be used in the proof of the following proposition.

Proposition 2. 1. *Let $\mathscr{A}$ be any von Neumann algebra, $\mathscr{I}$ any uniformly closed ideal in $\mathscr{A}$. If A is thin relative to $\mathscr{I}$, then A satisfies (H).*

Proof. Let $A = Z+K$, $Z \in \mathscr{Z}$, $K \in \mathscr{I}$. Note that $\|PAP - AP\| = \|(I-P)AP\| = \|(I-P)KP\|$, so it is enough to show $\lim_{P \in \mathscr{P}} \|(I-P)KP\| = 0$. Let $\varepsilon > 0$ be given. It suffices to find a $P_0 \in \mathscr{P}$ such that $Q \in \mathscr{P}$ and $Q > P_0$ implies $\|(I-Q)KQ\| < \varepsilon$. Then for any $P \in \mathscr{P}$, $P_0 \vee P \in \mathscr{P}$, $P_0 \vee P > P$, and if $Q \in \mathscr{P}$ with $Q > P_0 \vee P$, then $\|(I-Q)KQ\| < \varepsilon$.

By the theorem of Wils, choose $T\in\mathcal{I}$ with $P_0=\mathrm{rp}(T)\in\mathcal{I}$, and $\|T-K\|<\varepsilon$. Then

$$\|(I-P_0)K\| = \|(I-P_0)(T-K)\| < \varepsilon.$$

Now, if $Q\in\mathcal{P}$ and $Q>P_0$, then $I-Q \leqq I-P_0$, so

$$\|(I-Q)KQ\| \leqq \|(I-Q)K\| \leqq \|(I-P_0)K\| < \varepsilon.$$

Hence the proposition follows.

It is easy to see that the converse of Proposition 2. 1 is usually false if $\mathcal{I}$ is not weakly dense in $\mathcal{A}$. In this case, each $A\in\mathcal{A}$ with $A\mathcal{I}=\{0\}=\mathcal{I}A$ satisfies (H).

The techniques in the proof of the next proposition are adapted from those in [1], [5], and [6]. As in [1], we define for any $B\in\mathcal{B}(\mathcal{H})$, any projection $P\in\mathcal{B}(\mathcal{H})$.

$$\eta_B(P\mathcal{H}) = \sup_{x\in P\mathcal{H},\ \|x\|=1} \|Bx-(Bx,x)x\|.$$

Proposition 2. 2. *Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{I}$ a uniformly closed ideal in $\mathcal{A}$, and φ an irreducible representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with $\varphi(\mathcal{I})\neq 0$. Then for any $A\in\mathcal{A}$,*

$$\inf_{P\in\mathcal{P}} \eta_{\varphi(A)}(\varphi(1-P)\mathcal{H}) \leqq \limsup_{P\in\mathcal{P}} \|PA(I-P)\|.$$

Proof. Let $\varepsilon>0$ and $P_0\in\mathcal{P}$ be given. It suffices to show that for each unit vector x in $\varphi(1-P_0)\mathcal{H}$, there is a projection $Q\in\mathcal{P}$, $Q>P_0$, with

$$\|\varphi(A)x-(\varphi(A)x,x)x\| \leqq \|QA(I-Q)\|+\varepsilon.$$

For, we then have that

$$\eta_{\varphi(A)}(\varphi(I-P_0)\mathcal{H}) \leqq \sup_{Q>P_0} \|QA(I-Q)\|+\varepsilon,$$

and in particular, there is some $Q_0>P_0$ in $\mathcal{P}$ with

$$\eta_{\varphi(A)}(\varphi(I-P_0)\mathcal{H}) \leqq \|Q_0A(I-Q_0)\|+2\varepsilon.$$

From this it follows that

$$\inf_{P\in\mathcal{P}} \eta_{\varphi(A)}(\varphi(I-P)\mathcal{H}) \leqq \limsup_{P\in\mathcal{P}} \|PA(I-P)\|.$$

Fix $P\in\mathcal{P}$ and $\varepsilon>0$. Let x be an arbitrary fixed unit vector in $\varphi(I-P)\mathcal{H}$. (If $\varphi(I-P)\mathcal{H}=0$, we are done.) Set $y=\varphi(A)x-(\varphi(A)x,x)x$. By a theorem of Kadison [9, p. 274, Theorem 1] there is a self-adjoint operator $C\in\mathcal{I}$ such that $\varphi(C)$ is equal to the one-dimensional projection with range spanned by y at the two points x and y; that is, $\varphi(C)y=y$ and $\varphi(C)x=0$. By considering C^2, we may assume C is a positive operator.

Following the argument in [6, p. 61, Proposition 3. 1], we may assume $0\leqq C\leqq 1$. For, let f be the continuous real-valued function defined on the interval $[0, \|C\|+1]$

by $f\equiv 0$ on $\left[0, \frac{1}{2}\right]$, $f\equiv 1$ on $\left[\frac{3}{4}, \|C\|+1\right]$ and f linear between $\frac{1}{2}$ and $\frac{3}{4}$. Then f is a uniform limit of polynomials $\{p_n\}$ with real coefficients and with no constant terms. Thus if $p_n(t)=\Sigma a_m t^m$, then

$$\varphi(p_n(C))y = \Sigma a_m \varphi(C)^m y = p_n(I)y \quad \text{and} \quad \varphi(p_n(C)x)=0.$$

Since $p_n(I)=1$, we have

$$\varphi(f(C))y=\lim \varphi(p_n(C))y=y \quad \text{and} \quad \varphi(f(C))x=0.$$

Let $E(\lambda)$ be the spectral resolution of C, and set $E=E((\delta, 1])$. Then $E\in\mathscr{I}$ [2, p. 855, Lemma 4. 1], and $\delta E \leqq C \leqq E+\delta$. Thus

$$\|y\|^2 = \|\varphi(C)y\|^2 = (\varphi(C)y, y) \leqq ((\varphi(E)+\delta)y, y) = \|\varphi(E)y\|^2+\delta\|y\|^2.$$

Thus for sufficiently small $\delta>0$, we have $\|\varphi(E)y\| \geqq \|y\|-\varepsilon$. Furthermore,

$$\delta^2\|\varphi(E)x\|=(\varphi(\delta E)x, x)\leqq(\delta(C)x, x)=0.$$

Now set $Q = E\vee P$. Then $Q\in\mathscr{I}$,

$$\|\varphi(Q)y\| \geqq \|\varphi(E)y\| \geqq \|y\|-\varepsilon \quad \text{and} \quad \varphi(Q)x=0.$$

Thus,

$$\|QA(I-Q)\| \geqq \|\varphi(QA(I-Q))x\| = \|\varphi(QA)x\|$$
$$= \|\varphi(Q)(\varphi(A)x-(\varphi(A)x, x)x)+\varphi(Q)(\varphi(A)x, x)x\| = \|\varphi(Q)y\| \geqq \|y\|-\varepsilon.$$

So, $\|QA(I-Q)\|+\varepsilon \geqq \|\varphi(A)x-(\varphi(A)x, x)x\|$, and the proof is complete.

Note that the following theorem does not conflict with Proposition 2. 1, when $\mathscr{A}$ has non-trivial center $\mathscr{Z}=C(\Omega)$. The hypothesis that $\mathscr{I}$ contain a primitive ideal insures that $\mathscr{I}\cap\mathscr{Z}$ is some maximal ideal $\mu\in\Omega$. Hence for any $Z\in\mathscr{Z}$, we have $Z(\mu)\in\mathbf{C}$, and $Z-Z(\mu)\in\mathscr{I}$.

Theorem 2. 3. *Let $\mathscr{A}$ be a von Neumann algebra and $\mathscr{I}$ a uniformly closed ideal in $\mathscr{A}$ which properly contains a primitive ideal of $\mathscr{A}$. If $A\in\mathscr{A}$ satisfies (H), then $A = \lambda+K$, $K\in\mathscr{I}$, λ a scalar operator.*

Proof. Note that $\|PA^*(I-P)\| = \|PAP-AP\|$ and that $A^* = \lambda+K$ implies $A = \bar{\lambda}+K^*$. Hence it is equivalent to assume that $\lim_{P\in\mathscr{P}} \|PA(I-P)\| = 0$, and show that $A = \lambda+K$, λ a scalar, $K\in\mathscr{I}$.

Let φ be the irreducible representation of $\mathscr{A}$ on $\mathscr{H}$ whose kernel is the primitive ideal properly contained in $\mathscr{I}$. Thus $\varphi(\mathscr{I})\neq\{0\}$, and by Proposition 2. 2,

$$\inf_{P\in\mathscr{P}} \eta_{\varphi(A)}(\varphi(I-P)\mathscr{H}) = 0.$$

In a particular, there is a sequence of projections $\{P_n\} \subset \mathscr{P}$ with

$$\eta_n = \eta_{\varphi(A)}(\varphi(I-P_n)\mathscr{H}) < 1/n, \quad n=1, 2, \ldots;$$

and we may assume the sequence $\{P_n\}$ is increasing.

Following the proof of [1, p. 115 Theorem 1] we have that

$$W_{\varphi(A)}(\varphi(I-P_n)\mathscr{H}) = \{(\varphi(A)x, x) : x \in \varphi(I-P_n)\mathscr{H}, \|x\| = 1\}$$

is a nested sequence of convex sets. Furthermore, by [1, p. 114, Lemma 2. 2],

$$\text{diameter } W_{\varphi(A)}(\varphi(I-P_n)\mathscr{H}) \leqq 8\|\varphi(A)\|^{\frac{1}{2}}\eta_n.$$

Hence there is a unique complex number λ which is adherent to every $W_{\varphi(A)}(\varphi(I-P_n)\mathscr{H})$. Set $K = A-\lambda$, $\varphi(K) = \varphi(A)-\lambda$. Applying [1, p. 115, Lemma 2. 3], we have

$$\|\varphi(K)(I-\varphi(P_n))\|^2 \leqq 65\|\varphi(A)\|\eta_n.$$

Thus $\{\varphi(K)(I-\varphi(P_n))\}$ converges to zero. Since $\varphi(KP_n) \in \varphi(\mathscr{I})$ for each n, where $\varphi(\mathscr{I})$ is uniformly closed, we see that $\varphi(K) \in \varphi(\mathscr{I})$. But ker $(\varphi) \subset \mathscr{I}$ then implies that $K \in \mathscr{I}$. Since $A = K+\lambda$, the proof is complete.

Theorem 2. 4. *Let $\mathscr{A}$ be a von Neumann factor and $\mathscr{I}$ any uniformly closed ideal in $\mathscr{A}$. Then $A \in \mathscr{A}$ satisfies (H) if and only if $A = \lambda+K$, $K \in \mathscr{I}$, λ a scalar operator.*

Proof. Let Ψ be a non-zero irreducible representation of the C^*-algebra $\mathscr{I}$ on some Hilbert space $\mathscr{H}$. Let φ be an extension of Ψ to an irreducible representation of $\mathscr{A}$ on $\mathscr{H}$ [4, 2. 10. 2 and 2. 11. 3]. The uniformly closed ideals in the factor $\mathscr{A}$ are totally ordered [12]. Since the kernel of φ is such an ideal, $\mathscr{I} \nsubseteq \ker \varphi$ implies $\ker \varphi \subset \mathscr{I}$. Thus the theorem follows by Theorem 2. 3 and Proposition 2. 1.

Theorem 2. 5. *Let $\mathscr{A}$ be a semifinite, properly infinite von Neumann algebra, or a type III algebra with no σ-finite central projection. If $\mathscr{M}$ is a maximal ideal in $\mathscr{A}$, then $A \in \mathscr{A}$ is thin relative to $\mathscr{M}$ if and only if $A = \lambda+M$, λ a scalar and $M \in \mathscr{M}$.*

Proof. We may assume A is not a factor, by the preceding theorem. Let $\mathscr{J}$ denote the strong radical of $\mathscr{A}$; then $\mathscr{J} \neq \{0\}$. In fact, if $\mathscr{A}$ is semi-finite, $\mathscr{J}$ contains all the finite projections of $\mathscr{A}$ [6, p. 55—56 and p. 58, Proposition 2. 3]. Thus every projection in $\mathscr{A}$ dominates a projection in $\mathscr{J}$. This is also the case in the type III algebra of the assumed sort [6, pp. 56—57]. The ideal $\mathscr{M} \cap \mathscr{Z} = \zeta$ is maximal in $\mathscr{Z}$ [10]. Let $[\zeta]$ denote the ideal in $\mathscr{A}$ generated by ζ. Then $\mathscr{M} = \mathscr{J} + [\zeta]$, [6, p. 58, Proposition 2. 3], and $[\zeta]$ is primitive [7, p. 213, Theorem 4. 7]. Furthermore, $\mathscr{J} \notin [\zeta]$, so $\mathscr{M}$ properly contains $[\zeta]$ [6, p. 62]. Thus the result follows by Theorem 2. 3 and Proposition 2. 1.

3. A different approach yields partial results in the type I case.

Theorem 3. 1. *Let $\mathscr{A}$ be a type* I *von Neumann algebra such that $\mathscr{A} = \mathscr{B}(\mathscr{H})\oplus \oplus C(\Omega)$. Let $\mathscr{I}$ be a uniformly closed weakly dense ideal in $\mathscr{A}$ of the form $\mathscr{I} = \mathscr{B}(\mathscr{H})\oplus \oplus \mathscr{J}$, for some ideal $\mathscr{J}$ in $C(\Omega)$. Then $A\in\mathscr{A}$ is thin relative to $\mathscr{I}$ if and only if A satisfies (H).*

Proof. The ideal $\mathscr{J}$ is $\mathscr{J}=\{f\in C(\Omega): f|_\Gamma\equiv 0\}$, for some closed set $\Gamma\subset\Omega$. Since $\mathscr{I}$ is weakly dense, the interior of Γ is empty. Let $\{x_i\}$ be some orthonormal basis for $\mathscr{H}$. Then $A\in\mathscr{A}$ may be written $A=(a_{ij})$, $a_{ij}\in C(\Omega)$.

Suppose $A\in\mathscr{A}$ is not thin. We wish to show $\lim\limits_{P\in\mathscr{P}}\|PAP-AP\|\neq 0$. We claim it suffices to consider $A=(a_{ij})$ with some $a_{rs}\notin\mathscr{J}$, for $r\neq s$. For, suppose $a_{ij}\in\mathscr{J}$, all $i\neq j$. Then $A\neq Z+K$ any $Z\in\mathscr{Z}$, $K\in\mathscr{I}$ implies that for some $v\in\Gamma$, and some indices $r\neq s$, $a_{rr}(v)\neq a_{ss}(v)$. Let $U=(u_{ij})$ be the unitary in $\mathscr{A}$ given by $u_{rr}=u_{rs}=u_{ss}= =1/\sqrt{2}$, $u_{sr}=-1/\sqrt{2}$, and $u_{ii}=1$ if $i\neq r, s$; $u_{ij}=0$ otherwise (all these being constant functions on Ω). It is easy to compute that if $B=U^*AU=(b_{ij})$, then $b_{rs}(v)\neq 0$. Thus $b_{rs}\notin\mathscr{J}$. Observe that $\lim\limits_{P\in\mathscr{P}}\|PAP-AP\| = \lim\limits_{P\in\mathscr{P}}\|PBP-BP\|$, so it suffices to show that this latter limit is non-zero. Thus the claim is established.

Assume $a_{rs}\notin\mathscr{J}$, so fix $\mu\in\Gamma$ with $|a_{rs}(\mu)|>0$. Let $\delta>0$ be a number such that $|a_{rs}(v)|>\delta$, all $v\in V$ some open neighborhood of μ. Choose any $P\in\mathscr{P}$, $P=(p_{ij})$. We construct a projection $Q\in\mathscr{P}$, $Q>P$ with $\|QAQ-AQ\| > \delta/2$. This will suffice to show $\lim\limits_{P\in\mathscr{P}}\|PAP-AP\|\neq 0$.

There is an open neighborhood X of Γ in Ω with $p_{rr}(v)<\varepsilon$, $p_{ss}(v)<\varepsilon$, all $v\in X$. Since the interior of Γ is empty, $V\cap(X\setminus\Gamma)$ is a non-empty open set. Let W be a non-empty open and closed subset of $V\cap(X\setminus\Gamma)$. Define projections in $\mathscr{P}$: $E=(e_{ij})$, $F=(f_{ij})$ with $e_{ss}=\chi_W=f_{rr}$, and $e_{ij}=0=f_{ij}$ otherwise. Set $Q = P\vee E$, so $Q\in\mathscr{P}$ also. If $T=(t_{ij})\in\mathscr{A}$, denote $T(v)=\{t_{ij}(v)\}\in\mathscr{B}(\mathscr{H})$ for each $v\in\Omega$.

Then

$$\|PE\|^2=\sup_{v\in\Omega}\sup_{\substack{x\in E(v)\mathscr{H}\\ \|x\|=1}}(P(v)x,x)=\sup_{v\in W}(P(v)x_s,x_s)=\sup_{v\in W}p_{ss}(v)<\varepsilon.$$

Similarly, $\|PF\|^2<\varepsilon$, and $EF=0$.

Observe that $\|QF\|$ is also small. For, let $v\in\Omega$, and consider $f\in F(v)\mathscr{H}$, $q\in Q(v)\mathscr{H}$, $p\in P(v)\mathscr{H}$ and $e\in E(v)\mathscr{H}$. Then

$$\|Q(v)F(v)\|^2=\sup_{\|f\|=1}\|Q(v)f\|^2=\sup_{\|f\|=1}\sup_{\|q\|=1}|(q,f)|^2.$$

For an arbitrary fixed pair of such vectors q and f, we can write $q=\gamma_1 p+\gamma_2 e$ where p and e are unit vectors. Then we can find some unit vector g with $(e,g)=0$, such that $p=\nu_1 e+\nu_2 g$. Using the fact that $\|PE\|^2<\varepsilon$, $\|PF\|^2<\varepsilon$ and $EF=0$, a routine calculation shows that $|(g,f)|^2 < \varepsilon/1-\varepsilon^2$. Thus $\|QF^2\| = \sup_{v\in\Omega} \|Q(v)F(v)\|^2 <$ $< \varepsilon/1-\varepsilon^2$. Therefore,

$$\|(I-Q)AQ\| \geqq \|F(I-Q)AQE\| \geqq \|FAE\| - \|FQAE\| > \sup_{v\in\Omega} \|F(v)A(v)E(v)\| -$$

$$-(\varepsilon/1-\varepsilon^2)^{1/2}\|A\| = \sup_{v\in w} |a_{rs}(v)| - (\varepsilon/1-\varepsilon^2)^{1/2}\|A\| > \delta - (\varepsilon/1-\varepsilon^2)^{1/2}\|A\|.$$

For a sufficiently small choice of ε, $\|(I-Q)AQ\| > \delta/2$.

The converse follows by Proposition 2. 1, and the proof is complete.

Note that if $\mathscr{A}$ is a type I infinite algebra, and $\mathscr{M}$ a maximal ideal, then Theorem 2. 5 characterizes the thin operators relative to $\mathscr{M}$. If $\mathscr{I}$ is a finite intersection of maximal ideals in $\mathscr{A}$, it is not hard to show that an operator satisfying (H) is thin relative to $\mathscr{I}$. For example, if $\mathscr{H}$ is separable and $\mathscr{A} = \mathscr{B}(\mathscr{H})\oplus C(\Omega)$, then there is a finite set $\Gamma\subset\Omega$ with $\mathscr{I} = \{A\in\mathscr{A} : A(v) \text{ is compact, all } v\in\Gamma\}$.

References

[1] A. Brown and C. Pearcy, Structure of commutators of operators, *Ann. of Math.*, **82** (1965), 112—127.

[2] A. Brown, C. Pearcy and D. Topping, Commutators and the strong radical, *Duke Math. J.*, **35** (1968), 853—860.

[3] J. Dixmier, Sur certains espaces considérés par M. H. Stone, *Summa Brasil. Math.*, **2** (1951), 151—182.

[4] J. Dixmier, *Les C*-algèbres et leur représentations,* Gauthier—Villars (Paris, 1969).

[5] R. G. Douglas and C. Pearcy, A characterization of thin operators, *Acta Sci. Math.*, **29** (1968), 295—297.

[6] H. Halpern, Commutators in properly infinite von Neumann algebras, *Trans. Amer. Math. Soc.*, **139** (1969), 55—73.

[7] H. Halpern, Irreducible module homomorphisms of a von Neumann algebra into its center, *Trans. Amer. Math. Soc.*, **140** (1969), 195—221.

[8] H. Halpern, Commutators modulo the center in properly infinite von Neumann algebras, *Trans. Amer. Math. Soc.*, **150** (1970), 55—68.

[9] R. V. Kadison, Irreducible operator algebras, *Proc. Nat. Acad. Sci. U.S.A.*, **43** (1957), 273—276.

[10] Yosinao Misonou, On a weakly central operator algebra, *Tohoku Math. J.*, **4** (1952), 194—202.

[11] W. Wils, Two-sided ideals in W^*-algebras, *J. reine angew. Math.*, **244** (1970), 55—68.

[12] F. B. Wright, The ideals in a factor, *Ann. of Math.*, **68** (1958), 475—483.

UNIVERSITY OF TORONTO

(Received October 26, 1971; revised April 27, 1972)