On the invariant subspace lattice $1+\omega^*$

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Abstract: This paper has two parts. In the first one it is shown that a Banach algebra with identity whose lattice of closed left ideals is order isomorphic to $1+\omega^*$ with "dimension gaps" equal to one, from one ideal to the next one, is always a (commutative) Banach algebra of power series with one-point Gelfand spectrum. In the second one, the fact that the algebra $\mathfrak{L}(\mathfrak{X})$ of all bounded linear operators on a complex separable Banach space $\mathfrak{X}$ contains a subalgebra with the above mentioned characteristics, is used to show that $\mathfrak{L}(\mathfrak{X})$ can be generated by two elements.

1. Throughout this paper $\mathfrak{X}$ will denote a complex separable infinite dimensional Banach space. Let $\mathfrak{L}(\mathfrak{X})$ be the set of all operators in $\mathfrak{X}$. Here and in what follows, *operator* will mean *bounded linear map* (from $\mathfrak{X}$ into $\mathfrak{X}$); similarly, *algebra* and *subspace* will mean *weakly closed subalgebra of* $\mathfrak{L}(\mathfrak{X})$ *containing the identity* I of $\mathfrak{X}$ and *closed linear manifold*, respectively. For a given algebra $\mathfrak{A}$, Lat $\mathfrak{A}$ denotes the lattice of invariant (under every operator in $\mathfrak{A}$) subspaces of $\mathfrak{A}$. $\mathfrak{A}$ is called a *strictly cyclic algebra* (s. c. a.) if there exists a vector $x_0\in\mathfrak{X}$ such that

$$\mathfrak{X}=\mathfrak{A}x_0=\{Ax_0 : A\in\mathfrak{A}\}.$$

$\mathfrak{A}$ is separated by $x_0\in\mathfrak{X}$ if $A\in\mathfrak{A}$ and $Ax_0=0$ imply $A=0$. If $\mathfrak{A}x_0=\mathfrak{X}$ and x_0 separates points of $\mathfrak{A}$, then we shall say that $\mathfrak{A}$ is a separated s. c. a. and that x_0 is a separating s. c. vector for $\mathfrak{A}$. It is known that if $\mathfrak{A}$ has a separating s. c. vector x_0, then the map $A\to Ax_0$ from $\mathfrak{A}$ onto $\mathfrak{X}$ is an isomorphism of Banach spaces. By means of this map, $\mathfrak{X}$ can be identified with a Banach algebra $\mathfrak{B}$ with identity e; then $\mathfrak{A}$ is identified with $\mathfrak{B}_L$, the algebra of all left multiplications in $\mathfrak{B}$ by elements of $\mathfrak{B}$ (i.e., the *regular left representation* of $\mathfrak{B}$) and $\mathfrak{A}'$, the commutant of $\mathfrak{A}$ in $\mathfrak{L}(\mathfrak{X})$, is identified with $\mathfrak{B}_R$ the algebra of all right multiplications (or, the *regular right representation* of $\mathfrak{B}$) (see [2; 5; 6; 10]).

Let $\mathfrak{B}$ be a Banach algebra with identity and let $\mathfrak{B}_L$ be its regular left representation; the invariant subspaces of $\mathfrak{B}_L$ are, precisely, the closed left ideals of $\mathfrak{B}$. This justifies the following notation:

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Lat $\mathfrak{B}$ = Lat $\mathfrak{B}_L$ = {closed left ideals of $\mathfrak{B}$}.

The first part of this paper is devoted to proving that a class of Banach algebras with linearly ordered lattice are singly generated.

Theorem 1. *Let $\mathfrak{A}$ be a separated s. c. a. on $\mathfrak{X}$ and assume that*

(1) Lat $\mathfrak{A} = \{(0)\} \cup \{\mathfrak{M}_n\}_{n=0}^{\infty}$,

where

(2) $\mathfrak{X} = \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \cdots \supset \mathfrak{M}_n \supset \mathfrak{M}_{n+1} \supset, \ldots,$ *and*

(3) $\dim \mathfrak{M}_n/\mathfrak{M}_{n+1} = 1$, *for all* $n = 0, 1, 2, \ldots$.

Then there exists a quasi-nilpotent operator $T \in \mathfrak{A}$ such that

i) $\mathfrak{M}_n =$ *closure* $T^n(\mathfrak{X})$, $n = 0, 1, 2, \ldots$,

ii) $\mathfrak{A} = \mathfrak{A}_T =$ *strong closure of the polynomials in T, and*

iii) $\mathfrak{A}$ *is a Banach algebra of power series in the sense of Lorch and Shilow* (see [8; 10, p. 317; 12]). *In particular, $\mathfrak{A}$ is abelian; the Gelfand spectrum of $\mathfrak{A}$ consists of a single point.*

Remarks. (a) Lat $\mathfrak{A}$ is always complete; hence (1) and (2) imply $\bigcap_{n=0}^{\infty} \mathfrak{M}_n = (0)$.

(b) (2) says that Lat $\mathfrak{A}$ is order isomorphic to $1+\omega^*$ (where where ω is the first non-finite ordinal number). (3) says that the "dimensional gaps" are all equal to one. An invariant subspace (or closed left ideal) lattice satisfying (1), (2) and (3) will be denoted by: Lat $\mathfrak{A} \cong 1+\omega^*(dg = 1)$.

(c) It was shown in [6] that, if $\mathfrak{A}$ is a separated s. c. a., then the uniform and the strong operator topologies coincide on $\mathfrak{A}$. Hence, in Theorem 1, ii), "strong closure" is actually equivalent to "uniform closure".

Because of the previous identification of $\mathfrak{X}$ with a Banach algebra with identity, Theorem 1 can be rephrased as

Theorem 1′. *Let $\mathfrak{B}$ be a Banach algebra with identity e and assume that* Lat $\mathfrak{B} \cong 1+\omega^*(dg = 1)$. *Then $\mathfrak{B}$ is a Banach algebra of power series with a quasi-nilpotent generator t. The Gelfand spectrum of $\mathfrak{B}$ consists of a single point and the only non-zero closed left ideals of $\mathfrak{B}$ are those of the form* $\mathfrak{M}_n = \mathrm{cl}\,(t^n\mathfrak{B})$, $(n = 0, 1, 2, \ldots)$.

2. In what follows, $\mathfrak{B}$ will always denote a Banach algebra with identity e. The proof of Theorem 1 follows from a combination of Banach algebra methods and invariant subspace theory.

Lemma 1. *If* Lat $\mathfrak{B}$ *is linearly ordered, then every closed left ideal is a bilateral ideal.*

Lemma 2. *If* $\mathrm{Lat}\,\mathfrak{B} \cong 1+\omega^*(dg=1)$, *then* $\mathfrak{B}$ *has no zero divisors.*

Corollary 3. *Assume that* $\mathrm{Lat}\,\mathfrak{B} \cong 1+\omega^*(dg=1)$ *and let* $t\in\mathfrak{B}$, $t\neq 0$. *Then the map*

$$\sum_{k=0}^{N} c_k z^k \to \sum_{k=0}^{N} c_k t^k$$

from the polynomials in one indeterminate into $\mathfrak{B}$ *is one-to-one.*

Lemma 1 follows from a general fact about operator algebras: consider $\mathrm{Lat}\,\mathfrak{B} = \mathrm{Lat}\,\mathfrak{B}_L$ under the topology for invariant subspaces given in [7] (see also [1, 11]); since $\mathrm{Lat}\,\mathfrak{B}_L$ is linearly ordered, every point of $\mathrm{Lat}\,\mathfrak{B}_L$ is isolated. Therefore, ([7]) $\mathrm{Lat}\,\mathfrak{B}_L \subset \mathrm{Lat}\,(\mathfrak{B}_L)' = \mathrm{Lat}\,\mathfrak{B}_R$. Finally, observe that $\mathrm{Lat}\,\mathfrak{B}_R = \{$closed right ideals of $\mathfrak{B}\}$, from which the result follows.

Now assume that $\mathrm{Lat}\,\mathfrak{B} \cong 1+\omega^*(dg=1)$ and let $a, b\in\mathfrak{B}$, $a\neq 0$ and $ab=0$. Then $\mathfrak{M}=\{c\in\mathfrak{B}: cb=0\}=\ker R_b$ (R_b=right multiplication by $b\in\mathfrak{B}_R$) is a non-zero closed left ideal and therefore $\mathfrak{M}=\mathfrak{M}_k$ for some $k\geqq 0$, hence $\dim \mathfrak{B}/\mathfrak{M}_k = k<\infty$. It follows that rank $R_b \leqq k<\infty$.

On the other hand, closure range $R_b = \mathrm{cl}\,(R_b\mathfrak{B}) \in \mathrm{Lat}\,\mathfrak{B}_L$. Thus, either $\mathrm{cl}\,(R_b\mathfrak{B}) = (0)$ (and therefore $b=0$) or $\mathrm{cl}\,(R_b\mathfrak{B}) = \mathfrak{M}_h$, for some $h\geqq 0$. Since $\dim \mathrm{cl}\,(R_b\mathfrak{B}) = \mathrm{rank}\, R_b \leqq k<\infty$ and $\dim \mathfrak{M}_h$ is not finite, the second case must be ruled out. We conclude that $b=0$, and the proof of Lemma 2 is complete.

Finally, Corollary 3 is an easy consequence of Lemma 2.

Lemma 4. *Assume that* $\mathrm{Lat}\,\mathfrak{B} \cong 1+\omega^*(dg=1)$ *and let* $a\in\mathfrak{B}$. *Then*

i) $\sigma(a)$ (=the spectrum of a in $\mathfrak{B}$) *consists of a single point; moreover,* $\sigma(a) = \sigma(L_a) = \sigma(R_a)$, *where* $\sigma(L_a)$ ($\sigma(R_a)$, *respectively*) *denotes the spectrum of the left (right, respectively) multiplication by* a *as an operator on* $\mathfrak{B}$.

ii) a *is invertible in* $\mathfrak{B}$ *if and only if for some* $n\geqq 0$ *and some* $b\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$, $ab\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$.

Proof. i) Observe that $\mathfrak{B}$ has a *unique maximal bilateral ideal,* $\mathfrak{M}_1$, and that $\dim \mathfrak{B}/\mathfrak{M}_1 = 1$; hence, given $a\in\mathfrak{B}$, there exists a unique complex number $\lambda=\lambda(a)$ such that $a-\lambda e\in\mathfrak{M}_1$. Therefore, $\lambda\in\sigma(a)$.

If $\mu\neq\lambda$, then $\mathrm{cl}\,(a-\mu e)\mathfrak{B}$ is a closed left ideal of $\mathfrak{B}$, not contained in $\mathfrak{M}_1$. It follows that $\mathrm{cl}\,(a-\mu e)\mathfrak{B} = \mathfrak{B}$; then $(a-\mu e)\mathfrak{B}$ is a dense left ideal of $\mathfrak{B}$ and therefore (see [10, Chapter 1]) $(a-\mu e)$ has a right inverse b in $\mathfrak{B}$. Since, by Lemma 2, $\mathfrak{B}$ has no zero divisors, $a-\mu e\neq 0$ and $(a-\mu e)[e-b(a-\mu e)]=0$, we conclude that $b = (a-\mu e)^{-1}$, i.e. $\mu\notin\sigma(a)$. Therefore $\sigma(a)=\{\lambda(a)\}$.

The remaining statements follow from [6; 10] (in particular, a is invertible in $\mathfrak{B}$ if and only if L_a is invertible in $\mathfrak{L}(\mathfrak{B})$ if and only if R_a is invertible in $\mathfrak{L}(\mathfrak{B})$).

ii) If a is invertible, then $a\mathfrak{M}_n\subset\mathfrak{M}_n$ and $a^{-1}\mathfrak{M}_n\subset\mathfrak{M}_n$, for all $n\geqq 0$. It follows that $a\mathfrak{M}_n = a^{-1}\mathfrak{M}_n = \mathfrak{M}_n$ for all n. Therefore, $a(\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}) = \mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$, for all n.

Conversely, if $ab\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$ for some $b\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$ (and some $n\geqq 0$), then we can write $\mathfrak{M}_n=\mathfrak{M}_{n+1}\oplus\{\lambda b:\lambda\in\mathbf{C}\}$ and $ab=\lambda b+b'$, for some $\lambda\neq 0$ and some $b'\in\mathfrak{M}_{n+1}$. Hence, $(a-\lambda e)b=b'\in\mathfrak{M}_{n+1}$. If $(a-\lambda e)$ were invertible, then $b=(a-\lambda e)^{-1}b'$ would belong to $\mathfrak{M}_{n+1}$ (because $\mathfrak{M}_{n+1}$ is invariant under $\mathfrak{B}$!), contradicting our assumption. This proves that $a-\lambda e$ is not invertible in $\mathfrak{B}$. Now, i) implies that $\sigma(a)=\{\lambda\}$; since $\lambda\neq 0$, we conclude that a is invertible.

Proof of Theorem 1′. Let t be any element of $\mathfrak{M}_1\setminus\mathfrak{M}_2$.

Claim. $\mathfrak{B}$ coincides with the uniform closure of the polynomials in t.

Assume that, for each $n\geqq 0$, $t^n\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$. Then, since $\dim\mathfrak{M}_n/\mathfrak{M}_{n+1}=1$, for all n, it is not difficult to see that the finite linear combinations of the t^n's, $n=0, 1, 2, \dots$ (i.e. the polynomials in t) are uniformly dense in $\mathfrak{B}$.

Thus, in order to prove our claim, we only have to show that $t^n\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$, for $n=0, 1, 2, \dots$. We proceed by induction. Our choice of t implies that the above result is true for $n=0, 1$; let $m>1$ be the first index such that $t^m\notin\mathfrak{M}_m\setminus\mathfrak{M}_{m+1}$. Since $t^{m-1}\in\mathfrak{M}_{m-1}$, t^{m-1} is not invertible and Lemma 4 implies that $t^m=t^{m-1}t\in\mathfrak{M}_m$; thus, our hypothesis on t^m is equivalent to: $t^m\in\mathfrak{M}_{m+1}$. It follows that $\mathfrak{M}=\{\lambda t^{m-1}\}\oplus\oplus\mathfrak{M}_{m+1}\in\operatorname{Lat}L_t\setminus\operatorname{Lat}\mathfrak{B}$; therefore, there exists $a\in\mathfrak{B}$ such that $a\mathfrak{M}\not\subset\mathfrak{M}$.

Let $b\in\mathfrak{M}_m\setminus\mathfrak{M}_{m+1}$; it is not hard to see that a can be written (in a unique form) as

$$a=\lambda_0 e+\lambda_1 t+\dots+\lambda_{m-1}t^{m-1}+\lambda_m b+a',$$

where $\lambda_0, \dots, \lambda_m\in C$ and $a'\in\mathfrak{M}_{m+1}$.

Now, the invariance of $\mathfrak{M}$ under L_t implies that $t^n\mathfrak{M}\subset\mathfrak{M}$, for all $n\geqq 0$. On the other hand, since $a'\in\mathfrak{M}_{m+1}$ and $\mathfrak{M}_{m+1}$ is a bilateral ideal (use Lemma 1), it is not hard to see that

$$a'\mathfrak{M}\subset a'\mathfrak{B}\subset\mathfrak{M}_{m+1}\subset\mathfrak{M}.$$

Thus, $a\mathfrak{M}\not\subset\mathfrak{M}$ if and only if $\lambda_m\neq 0$ and $b\mathfrak{M}\neq\mathfrak{M}$. Moreover, $b\mathfrak{M}_{m+1}\subset\mathfrak{M}_{m+1}$; therefore, $a\mathfrak{M}\not\subset\mathfrak{M}$ is equivalent to: $bt^{m-1}\notin\mathfrak{M}$. But this last statement cannot be true. In fact, since b is not invertible and $t^{m-1}\in\mathfrak{M}_{m-1}\setminus\mathfrak{M}_m$, it follows from Lemma 4 that $bt^{m-1}\in\mathfrak{M}_m$, i.e., $bt^{m-1}=\lambda b+b'$, for some $\lambda\in C$ and some $b'\in\mathfrak{M}_{m+1}$. Now if $\lambda=0$, then $bt^{m-1}\in\mathfrak{M}_{m+1}\subset\mathfrak{M}$, contradicting our assumption. If $\lambda\neq 0$, then $b(t^{m-1}-\lambda e)=b'\in\mathfrak{M}_{m+1}$ and (by Lemma 2) $(t^{m-1}-\lambda e)$ is invertible in $\mathfrak{B}$; hence $b=b'(t^{m-1}-\lambda e)^{-1}\in\mathfrak{M}_{m+1}$ (here we are using the fact that $\mathfrak{M}_{m+1}$ is a bilateral ideal, i.e., Lemma 1), again we obtain a contradiction.

We conclude that $t^n\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$, for all $n\geqq 0$ and $\mathfrak{B}$ is the uniform closure of the polynomials in t.

By Lemma 4, t is quasi-nilpotent. To complete the proof we only have to show that $\mathfrak{B}$ is a Banach algebra of power series in t. This is also clear: observe that, for each fixed $n \geqq 0$, $\mathfrak{B}$ can be written (in a unique fashion) as the direct sum

$$\mathfrak{B} = \{\lambda_0 e\} \oplus \{\lambda_1 t\} \oplus \cdots \oplus \{\lambda_n t^n\} \oplus \mathfrak{M}_{n+1}$$

with complex $\lambda_0, \ldots, \lambda_n$.

Define γ_k by $\gamma_k(t^n)=1$, $\gamma_k(t^k)=0$ for $n = 0, 1, \ldots, k-1$ and $\gamma_k(\mathfrak{M}_{k+1})=0$. It is clear that γ_k is a continuous linear functional on $\mathfrak{B}$ and that every element a of $\mathfrak{B}$ can be written as a (unique!) formal power series in t: $a = \sum_{k=0}^{\infty} \gamma_k(a) t^k$. Since t generates $\mathfrak{B}$ and t is quasi-nilpotent, it is not hard to infer that the only non-zero continuous multiplicative functional on $\mathfrak{B}$ is γ_0; i.e. Gelfand's spectrum of $\mathfrak{B}$ is a single point.

It is completely apparent that $\mathfrak{M}_n = \mathrm{cl}\,(t^n \mathfrak{B})$; $n=0, 1, 2, \ldots$. The proof is complete now.

3. Generators of $\mathfrak{L}(\mathfrak{X})$. Recently, S. GRABINER ([3; 4]) showed that, for any $\mathfrak{X}$ satisfying our requirements it is possible to construct a chain $\mathfrak{J} \cong 1+\omega^*(dg = 1)$ of subspaces and a nuclear operator T such that

(4) T is a quasi-nilpotent; $\mathfrak{A}_T$ is a separated s. c. a., and $\mathrm{Lat}\, T = \mathfrak{J}$.

We are indebted to Professor GRABINER for sending us his unpublished paper [4] and to M. IMINA for several helpful discussions. We shall use Grabiner's result to prove that $\mathfrak{L}(\mathfrak{X})$ is always generated by two elements. In fact, we have the following:

Theorem 2. *Let $L \in \mathfrak{L}(\mathfrak{X})$, $L \neq \lambda I$ (for all $\lambda \in C$). Then there exists $T \in \mathfrak{L}(\mathfrak{X})$ such that $\mathfrak{L}(\mathfrak{X}) = \mathfrak{A}(T, L)$, the strong closure of the polynomials in T and L.*

The construction of a chain $\mathfrak{J} \cong 1+\omega^*(dg=1)$ of subspaces of $\mathfrak{X}$ is standard. This is equivalent to finding a sequence $\{\beta_n\}_{n=0}^{\infty} \subset \mathfrak{X}^*$ (the topological dual of $\mathfrak{X}$) such that the β_n' s are linearly independent, $\mathfrak{M}_n = \bigcap_{k=0}^{n-1} \ker \beta_k$ ($n=1, 2, 3, \ldots$) and $\bigcap_{k=0}^{\infty} \ker \beta_k = (0)$ (i.e., $\{\beta_n\}$ is *total* on $\mathfrak{X}$). Then, if $\mathfrak{M}_0 = \mathfrak{X}$, the lattice $\mathfrak{J} = \{(0)\} \cup \{\mathfrak{M}_n\}$ satisfies our requirements; furthermore, $\beta_0 (\neq 0)$ can be arbitrarily chosen in $\mathfrak{X}^*$. We shall need two auxiliary lemmas; the first one says that a "small perturbation" of $\{\beta_n\}$ provides a new lattice, $\mathfrak{J}'$ with similar characteristics.

Lemma 5. *Let $\{\beta_n\}_{n=0}^{\infty}$ be a total set of linearly independent functionals such that $\bigcap_{n=0}^{\infty} \ker \beta_n = (0)$ and $\|\beta_n\| \geqq 1$ for all $n \geqq 0$. If $0 \leqq \varepsilon_n \leqq (2\|\beta_n\|)^{-1}$, then $\{\beta_n' = \beta_n + \varepsilon_{n+1} \beta_{n+1}\}_{n=0}^{\infty}$ is also a linearly independent total set of linear functionals on $\mathfrak{X}$.*

Proof. Let $x \in \mathfrak{X}$ and assume that $\beta_n'(x)=0$, for all n. Then $\beta_n(x) = -\varepsilon_{n+1}\beta_{n+1}(x)$ and, by induction on k, we have

$$\beta_n(x)=(-1)^k \varepsilon_{n+1}\varepsilon_{n+2}\dots\varepsilon_{n+k-1}\varepsilon_{n+k}\beta_{n+k}(x), \qquad k=2, 3, \dots .$$

Hence

$$|\beta_n(x)| \leqq 2^{-(k-1)}\|\varepsilon_{n+k}\beta_{n+k}\|\,\|x\| \leqq 2^{-(k-1)}\|x\| \to 0, \quad \text{as} \quad k\to\infty.$$

Therefore $\beta_n(x)=0$ for all $n \geqq 0$. Since $\{\beta_n\}_{n=0}^{\infty}$ is total, it follows that $x=0$; i.e., $\{\beta_n'\}$ is also a total set. The linear independence of $\{\beta_n'\}$ is also clear.

Lemma 6. *Let $L \in \mathfrak{L}(\mathfrak{X})$, $L \neq \lambda L$. Then there exists a lattice $\mathfrak{J} \cong 1+\omega^*(dg = 1)$ such that* $\operatorname{Lat} L \cap \mathfrak{J} = \{(0), \mathfrak{X}\}$.

Proof. If every $\beta \in \mathfrak{X}^*$ is an eigenvector of L^*, then $L^*=\lambda I^*$ (I^*=the identity operator on $\mathfrak{X}^*$), for some $\lambda \in C$, and therefore $L=\lambda I$, contradicting our hypothesis. Therefore, we can find a vector $\beta_0 \in \mathfrak{X}^*$, $\|\beta_0\| \geqq 1$. which is not an eigenvector of L^*; equivalently, $\ker \beta_0 = \mathfrak{M}_1 \notin \operatorname{Lat} L$.

Complete $\{\beta_0\}$ to a total set $\{\beta_n\}_{n=0}^{\infty}$ of linearly independent functionals of norm $\geqq 1$ and set $\beta_n' = \beta_n + \varepsilon_{n+1}\beta_{n+1}$, where $\varepsilon_n = (2\|\beta_n\|)^{-1}$, $n=0, 1, 2, \dots$.

Write $\mathfrak{L}=(0^*)$, $\mathfrak{L}_1=\{\lambda\beta_0 : \lambda \in C\}$. Now we proceed by induction; assume that $\mathfrak{L}_n$ has been defined in such a way that $\mathfrak{L}_n \subset \text{lin span}\,[\{\beta_0, \dots, \beta_n\}$, $\mathfrak{L}_n \notin \operatorname{Lat} L^*$ and $\beta_{m-1} \notin \mathfrak{L}_{m-1}$ for $n=0, 1, \dots, m-1$. If both $\mathfrak{L}_{m-1} \oplus \{\lambda\beta_{m-1}\}$ and $L_{m-1} \oplus \{\lambda\beta_{m-1}'\}$ belong to $\operatorname{Lat} L^*$, then $\mathfrak{L}_{m-1} = (\mathfrak{L}_{m-1} \oplus \{\lambda\beta_{m-1}\}) \cap (\mathfrak{L}_{m-1} \oplus \{\lambda\beta_{m-1}'\}) \in \operatorname{Lat} L^*$, contradicting our inductive hypothesis. Hence, either $\mathfrak{L}_{m-1} \oplus \{\lambda\beta_{m-1}\} \notin \operatorname{Lat} L^*$ or $\mathfrak{L}_{m-1} \oplus \oplus \{\lambda\beta_{m-1}\} \in \operatorname{Lat} L^*$ and $\mathfrak{L}_{m-1} \oplus \{\lambda\beta_{m-1}'\} \notin \operatorname{Lat} L^*$. In the first case, we write $\mathfrak{L}_m = \mathfrak{L}_{m-1} \oplus \oplus \{\lambda\beta_{m-1}\}$ and $\beta_{m-1}'' = \beta_{m-1}$; in the second one, we take $\mathfrak{L}_m = \mathfrak{L}_{m-1} \oplus \{\lambda\beta_{m-1}'\}$ and $\beta_{m-1}'' = \beta_{m-1}'$.

Thus we have constructed a sequence $\{\mathfrak{L}_n\}_{n=0}^{\infty}$ of subspaces such that 1) $\mathfrak{L}_n \subset \mathfrak{L}_{n+1}$; 2) $\dim \mathfrak{L}_n = n$, $n=0, 1, \dots$, and 3) $\mathfrak{L}_n \notin \operatorname{Lat} L^*$. It is not hard to see using Lemma 5 that the lattice $\mathfrak{J} = \{(0)\} \cup \{\mathfrak{M}_n\}_{n=0}^{\infty}$ where $\mathfrak{M}_n = \mathfrak{L}_n^{\perp} = \bigcap_{k=0}^{n-1} \ker \beta_k''$ satisfies our requirements, i.e. $\mathfrak{J} \cong 1+\omega^*(dg = 1)$ and $\operatorname{Lat} L \cap \mathfrak{J} = \{(0), \mathfrak{X}\}$.

Now we are in a position to prove Theorem 2. Let $L \in \mathfrak{L}(\mathfrak{X})$, $L \neq \lambda I$ and let $\mathfrak{J}$ be chosen as in Lemma 6. Using Grabiner's result, we construct an operator $T \in \mathfrak{L}(\mathfrak{X})$ satisfying (4). It is clear that $\mathfrak{A}(T, L)$ is a *transitive* subalgebra of $\mathfrak{L}(\mathfrak{X})$ (i.e., it has no non-trivial invariant subspaces); in fact,

$$\operatorname{Lat} \mathfrak{A}(T, L) = \operatorname{Lat} L \cap \operatorname{Lat} T = \operatorname{Lat} L \cap \mathfrak{J} = \{(0), \mathfrak{X}\}.$$

On the other hand, $\mathfrak{A}_T$ is a strictly cyclic subalgebra of $\mathfrak{A}(T, L)$. These two properties of $\mathfrak{A}(T, L)$ and the results of [2; 5] imply that $\mathfrak{A}(T, L) = \mathfrak{L}(\mathfrak{X})$.

Remarks. a) In [9], H. Radjavi and P. Rosenthal proved that if L is any operator in the complex (or real) separable Hilbert space $\mathfrak{X}$ such that $L \neq \lambda I$ ($\lambda \in C$), then there exists a compact hermitian operator $H \in \mathfrak{L}(\mathfrak{X})$ such that $\mathfrak{L}(\mathfrak{X}) = \mathfrak{A}(L, H)$. Thus, Theorem 2 can be considered as a result for Banach spaces which is analogous to the above one.

b) Let $\mathfrak{X}$ be as usual and let β be a non-zero continuous linear functional on $\mathfrak{X}$. For each $z \in \ker \beta$ and each $\lambda \in C$, define $A_{z,\lambda} y = \lambda y + \beta(y) z$ and $\mathfrak{A} = \{A_{z,\lambda} : z \in \ker \beta,\ \lambda \in C\}$; it is not hard to check (see [2]) that $\mathfrak{A}$ is an abelian separated s. c. a. (x_0 is a separating s. c. vector for $\mathfrak{A}$ if and only if $\beta(x_0) \neq 0$). A straightforward computation shows that the subalgebra generated by $\{A_{z_\nu}, \lambda_\nu : \nu \in \Sigma\}$ is equal to $\{A_{z,\lambda} : \lambda \in C,\ z \in \text{closed lin span } [z_\nu : \nu \in \Sigma]\}$; in particular, $\mathfrak{A}$ cannot be finitely generated.

With minor modifications of the same example it is not hard to show that $\mathfrak{L}(\mathfrak{X})$ contains, for each n ($n = 2, 3, 4, \ldots, \aleph_0$), an abelian separated s. c. a. $\mathfrak{A}_n$ which can be generated by n operators, but no set of $n-1$ operators generates $\mathfrak{A}_n$. Thus, the statement of Theorem 2 cannot be extended to arbitrary subalgebras of $\mathfrak{L}(\mathfrak{X})$.

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