On the structure of intertwining operators

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A theorem proved in our previous paper [6] asserts that every operator X intertwining two contractions, T_1 and T_2, can be lifted, without increasing norm, to an operator Y intertwining their minimal isometric dilations, V_1 and V_2. This theorem allows a study of the structure of such operators: this will be done, in a purely geometric manner, in Sec. 1. Then, in Sec. 2, the results of Sec. 1 will be reformulated for the case where the contractions T_k $(k=1, 2)$ are completely non-unitary and appear in their functional models $S(\Theta_k)$.

Particular interest lies with intertwining operators X which have a (bounded) inverse and thus establish similarity between T_1 and T_2. We obtain in this way among others a criterion for a contraction to be similar to some isometry (and a new proof of the known criterion for a contraction to be similar to some unitary operator). The main criteria of similarity concern two contractions, arbitrary or completely non-unitary, in the latter case given by their functional models $S(\Theta_k)$ $(k=1, 2)$. One of these criteria, stated in Sec. 3, is particularly interesting since it only involves relations between analytic functions and a certain equidimensionality condition. This criterion generalizes a former result of KRIETE [3], which concerns operators $S(\Theta_k)$ with scalar valued contractive analytic functions Θ_k.

Sec. 4 is devoted to problems concerning the commutant $(T)'$ of a c.n.u. contraction $T=S(\Theta)$. Namely, a necessary condition is given for the characteristic function $\Theta(z)$ in order that $(T)'$ should consist of functions $\varphi(T)$, φ belonging to the Nevanlinna class N_T. Moreover, it is proved that if $T=S(\Theta)$ with scalar Θ, then $(T)'$ is always commutative, with the exception of a single case.

Finally, in Sec. 5, functions $u(T)$ (with $u\in H^\infty$) of a c.n.u. contraction T are considered, and a criterion is established for $u(T)$ to be boundedly invertible; this criterion generalizes an earlier result of FUHRMANN [1].

1. Contractions of general type

1. For any two operators on Hilbert spaces, say T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$, denote by $\mathscr{I}(T_1, T_2)$ the set of (linear, bounded) operators $X: \mathfrak{H}_1 \to \mathfrak{H}_2$ such that

$$(1.1) \qquad T_2 X = X T_1.$$

If T_1 and T_2 are *contractions*, and V_1 and V_2 are their minimal isometric dilations (cf. [5], Chapter I) acting on the spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, then let $\mathscr{I}^+(T_1, T_2)$ denote the set of operators $Y: \mathfrak{K}_1 \to \mathfrak{K}_2$ belonging to $\mathscr{I}(V_1, V_2)$ and satisfying the additional condition

$$(1.2) \qquad P_2 Y P_1 = P_2 Y,$$

where P_i denotes the orthogonal projection from $\mathfrak{K}_i$ onto $\mathfrak{H}_i$.

Clearly $\mathscr{I}(T_1, T_2)$ and $\mathscr{I}^+(T_1, T_2)$ are subspaces of the Banach spaces of all operators from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ and from $\mathfrak{K}_1$ into $\mathfrak{K}_2$, respectively.

As T_i and V_i are connected by the relation

$$(1.3) \qquad T_i P_i = P_i V_i,$$

condition $Y \in \mathscr{I}^+(T_1, T_2)$ implies

$$T_2 P_2 Y = P_2 V_2 Y = P_2 Y P_1 V_1 = P_2 Y T_1 P_1,$$

i.e. the operator

$$(1.4) \qquad X = P_2 Y | \mathfrak{H}_1$$

belongs to $\mathscr{I}(T_1, T_2)$. Thus the transformation $Y \to X$ defined by (1.4) is a map

$$\pi_{12}: \mathscr{I}^+(T_1, T_2) \to \mathscr{I}(T_1, T_2),$$

which is obviously linear and does not increase norm (i.e., $\|X\| \leqq \|Y\|$). Observe that on account of (1.2) relation (1.4) implies

$$(1.5) \qquad X P_1 = P_2 Y;$$

conversely, relation (1.5) implies both (1.2) and (1.4).

The "Lifting Theorem" for intertwining operators (see [6], or [5], Sec. II. 2) asserts that the above map π_{12} is actually *onto*, moreover for every $X \in \mathscr{I}(T_1, T_2)$ there exists at least one $Y \in \mathscr{I}^+(T_1, T_2)$ satisfying (1.5) and such that $\|X\| = \|Y\|$.

The aim of this paper is a further analysis of this map π_{12}, and some of its applications.

To begin with, let us state the following immediate consequence of relation (1.5):

Multiplication Property: If T_1, T_2, T_3 are any three contractions and if

$$Y \in \mathscr{I}^+(T_1, T_2), \quad Z \in \mathscr{I}^+(T_2, T_3),$$

then

(1.6) $$ZY \in \mathscr{I}^+(T_1, T_3) \quad \text{and} \quad \pi_{13}(ZY) = \pi_{23}(Z)\pi_{12}(Y).$$

Also note that

(1.7) $$I_{\mathfrak{K}_1} \in \mathscr{I}^+(T_1, T_1) \quad \text{and} \quad \pi_{11}(I_{\mathfrak{K}_1}) = I_{\mathfrak{H}_1}.$$

2. Let us return to the case of two contractions, T_1 and T_2. Consider the Wold decomposition of the space $\mathfrak{K}_i$ generated by the minimal isometric dilation V_i of T_i $(i=1, 2)$, i.e. let

(1.8) $$\mathfrak{K}_i = \mathfrak{S}_{*i} \oplus \mathfrak{R}_i, \quad \text{where} \quad \mathfrak{R}_i = \bigcap_{n=0}^{\infty} V_i^n \mathfrak{K}_i;$$

the subspaces $\mathfrak{S}_{*i}$ and $\mathfrak{R}_i$ reduce V_i respectively to its unilateral shift part S_{*i} and its unitary part R_i (one of these subspaces may be missing, i.e. equal $\{0\}$). Then we have for any $Y \in \mathscr{I}(V_1, V_2)$:

$$Y\mathfrak{R}_1 = \bigcap_{n=0}^{\infty} YV_1^n \mathfrak{K}_1 = \bigcap_{n=0}^{\infty} V_2^n Y \mathfrak{K}_1 \subset \bigcap_{n=0}^{\infty} V_2^n \mathfrak{K}_2 = \mathfrak{R}_2.$$

Therefore, if both $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are decomposed according to (1.8) the operator Y will be represented by a matrix

(1.9) $$Y = \begin{bmatrix} A_* & 0 \\ B & C \end{bmatrix},$$

where

(1.10) $$A_* \in \mathscr{I}(S_{*1}, S_{*2}), \quad B \in \mathscr{I}(S_{*1}, R_2), \quad C \in \mathscr{I}(R_1, R_2).$$

Clearly, conditions (1.10) are also sufficient for Y to belong to $\mathscr{I}(V_1, V_2)$.

Now we are going to analyse condition (1.2). To this end first recall (*cf.* [5], Sec. II. 2) that the subspace

(1.11) $$\mathfrak{S}_i = \mathfrak{K}_i \ominus \mathfrak{H}_i$$

is invariant for V_i and that

(1.12) $$S_i = V_i | \mathfrak{S}_i$$

is a unilateral shift. (It may happen that $\mathfrak{S}_i = \{0\}$: this is the case if T_i itself is an isometry.)

Introduce the operators

(1.13) $$\left.\begin{matrix} \hat{\Theta}_i \\ \hat{\Delta}_i \end{matrix}\right\} = \text{orthogonal projection of } \mathfrak{S}_i \text{ into} \left\{\begin{matrix} \mathfrak{S}_{*i} \\ \mathfrak{R}_i \end{matrix}\right. .$$

As $\mathfrak{S}_{*i}$ and $\mathfrak{R}_i$ are reducing subspaces for the isometry V_i we obviously have

(1.14) $$\hat{\Theta}_i \in \mathscr{I}(S_i, S_{*i}), \quad \hat{\Delta}_i \in \mathscr{I}(S_i, R_i).$$

Condition (1. 2) means that Y transforms $\mathfrak{S}_1$ into $\mathfrak{S}_2$. Hence we infer that an operator $Y \in \mathscr{I}(V_1, V_2)$ satisfies condition (1. 2) if and only if $A = Y|\mathfrak{S}_1$ belongs to $\mathscr{I}(S_1, S_2)$. Using for Y the matrix form (1. 9) and for $x_i \in \mathfrak{S}_i$ the column vector representation

$$x_i = \begin{bmatrix} \hat{\Theta}_i x_i \\ \hat{\Delta}_i x_i \end{bmatrix} \quad (i=1, 2),$$

and comparing the corresponding components we arrive at the following result:

Lemma 1. 1. *The operator Y with the matrix* (1. 9) *belongs to $\mathscr{I}^+(T_1, T_2)$ if and only if its entries satisfy conditions* (1. 10) *and*

$$A_* \hat{\Theta}_1 = \hat{\Theta}_2 A, \quad B\hat{\Theta}_1 + C\hat{\Delta}_1 = \hat{\Delta}_2 A, \tag{1. 15}$$

with some operator

$$A \in \mathscr{I}(S_1, S_2). \tag{1. 16}$$

3. Consider a $Y \in \mathscr{I}^+(T_1, T_2)$ for which $\pi_{12}(Y)=0$, i.e. $Y\mathfrak{H}_1 \subset \mathfrak{H}_2^\perp (=\mathfrak{S}_2)$. As by virtue of (1. 2) we also have $Y\mathfrak{S}_1 \subset \mathfrak{S}_2$, condition $Y\mathfrak{H}_1 \subset \mathfrak{S}_2$ is equivalent to the condition $Y\mathfrak{R}_1 \subset \mathfrak{S}_2$. Hence we infer first that

$$Y\mathfrak{R}_1 = \bigcap_{n=0}^{\infty} YV_1^n \mathfrak{R}_1 = \bigcap_{n=0}^{\infty} V_2^n Y\mathfrak{R}_1 \subset \bigcap_{n=0}^{\infty} V_2^n \mathfrak{S}_2 = \{0\}$$

(the latter equation holds because $V_2|\mathfrak{S}_2 (=S_2)$ is a unilateral shift); as a consequence we have $C=0$. Next, $Y\mathfrak{R}_1 \subset \mathfrak{S}_2$ also implies $Y\mathfrak{S}_{*1} \subset \mathfrak{S}_2$, and hence we deduce that the operator $D = Y|\mathfrak{S}_{*1}$ belongs to $\mathscr{I}(S_{*1}, S_2)$. Therefore we have for $x \in \mathfrak{S}_{*1}$

$$\begin{bmatrix} A_* & 0 \\ B & C \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = Yx = Dx = \begin{bmatrix} \hat{\Theta}_2 Dx \\ \hat{\Delta}_2 Dx \end{bmatrix}, \text{ i. e. } A_* = \hat{\Theta}_2 D, \quad B = \hat{\Delta}_2 D.$$

Conversely, one easily verifies that if D is any operator satisfying

$$D \in \mathscr{I}(S_{*1}, S_2) \tag{1. 17}$$

then the operators defined by

$$A = D\hat{\Theta}_1, \quad A_* = \hat{\Theta}_2 D, \quad B = \hat{\Delta}_2 D, \quad C = 0 \tag{1. 18}$$

satisfy conditions (1. 10), (1. 15), and (1. 16), and therefore the corresponding operator $Y = \begin{bmatrix} A_* & 0 \\ B & C \end{bmatrix}$ belongs to $\mathscr{I}^+(T_1, T_2)$. Moreover we have then

$$Y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \hat{\Theta}_2 Dx \\ \hat{\Delta}_2 Dx \end{bmatrix} = Dx \in \mathfrak{S}_2 \quad \text{for} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathfrak{R}_1,$$

and therefore

$$\pi_{12}(Y) = 0.$$

Thus we have proved:

Lemma 1. 2. *The general form of an operator* $Y \in \mathscr{I}^+(T_1, T_2)$ *satisfying* $\pi_{12}(Y)=0$ *is*

$$Y=\begin{bmatrix}\hat{\Theta}_2 D & 0\\ \hat{\Delta}_2 D & 0\end{bmatrix} \quad \textit{with arbitrary} \quad D \in \mathscr{I}(S_{*1}, S_2).$$

4. Suppose we have

$$Y=\begin{bmatrix}A_* & 0\\ B & C\end{bmatrix} \in \mathscr{I}^+(T_1, T_2) \quad \text{and} \quad Y'=\begin{bmatrix}A'_* & 0\\ B' & C'\end{bmatrix} \in \mathscr{I}^+(T_2, T_1).$$

Let $X=\pi_{12}(Y)$, $X'=\pi_{21}(Y')$. From the multiplication property (1. 6) and from (1. 7) we deduce that X and X' are inverse to each other if and only if

$$\pi_{11}(I_{\mathfrak{K}_1}-Y'Y)=0 \quad \text{and} \quad \pi_{22}(I_{\mathfrak{K}_2}-YY')=0.$$

On account of Lemmas 1. 1 and 1. 2 these two conditions in turn are equivalent to the condition that there exist operators

$$D \in \mathscr{I}(S_{*1}, S_1), \quad D' \in \mathscr{I}(S_{*2}, S_2)$$

satisfying the equations

$$\begin{bmatrix}I_{\mathfrak{S}_{*1}}-A'_*A_* & 0\\ -B'A_*-C'B & I_{\mathfrak{R}_1}-C'C\end{bmatrix}=\begin{bmatrix}\hat{\Theta}_1 D & 0\\ \hat{\Delta}_1 D & 0\end{bmatrix},$$

$$\begin{bmatrix}I_{\mathfrak{S}_{*2}}-A_*A'_* & 0\\ -BA'_*-CB' & I_{\mathfrak{R}_2}-CC'\end{bmatrix}=\begin{bmatrix}\hat{\Theta}_2 D' & 0\\ \hat{\Delta}_2 D' & 0\end{bmatrix};$$

thus in particular $C'=C^{-1}$. Since $C \in \mathscr{I}(R_1, R_2)$, this implies that the unitary operators R_1, R_2 are similar, and therefore unitarily equivalent.

Note that the existence of a boundedly invertible X in $\mathscr{I}(T_1, T_2)$ means that T_1 and T_2 are similar. Thus, also using Lemma 1. 2, we can summarize our results as follows:

Theorem 1. 3. *A* necessary *condition for the contractions* T_1 *and* T_2 *to be similar is that the unitary parts* R_1, R_2 *of their minimal isometric dilations be unitarily equivalent.* Necessary *and* sufficient *is the existence of operators*

$$(\sigma)\begin{cases}(\sigma_1)\begin{cases}A_* \in \mathscr{I}(S_{*1}, S_{*2}), & A \in \mathscr{I}(S_1, S_2), \quad D \in \mathscr{I}(S_{*1}, S_1)\\ A'_* \in \mathscr{I}(S_{*2}, S_{*1}), & A' \in \mathscr{I}(S_2, S_1), \quad D' \in \mathscr{I}(S_{*2}, S_2),\end{cases}\\[2ex] (\sigma_2)\begin{cases}B \in \mathscr{I}(S_{*1}, R_2), & C \in \mathscr{I}(R_1, R_2),\\ B' \in \mathscr{I}(S_{*2}, R_1), & C' \in \mathscr{I}(R_2, R_1)\end{cases}\end{cases}$$

satisfying the conditions

$$
\begin{array}{llll}
(\alpha) & A_*\hat{\Theta}_1=\hat{\Theta}_2 A, & (\beta) & B\hat{\Theta}_1+C\hat{\Delta}_1=\hat{\Delta}_2 A,\\
(\alpha') & A'_*\hat{\Theta}_2=\hat{\Theta}_1 A', & (\beta') & B'\hat{\Theta}_2+C'\hat{\Delta}_2=\hat{\Delta}_1 A',\\
(\gamma_*) & A'_*A_*+\hat{\Theta}_1 D=I_{\mathfrak{E}_{*1}}, & (\delta) & B'A_*+C'B=-\hat{\Delta}_1 D,\\
(\gamma'_*) & A_*A'_*+\hat{\Theta}_2 D'=I_{\mathfrak{E}_{*2}}, & (\delta') & BA'_*+CB'=-\hat{\Delta}_2 D',\\
(\varepsilon) & C'=C^{-1}. & &
\end{array}
$$

5. From conditions (α)—(ε) we deduce some further ones. Namely we have

$$\hat{\Theta}_1(A'A+D\hat{\Theta}_1-I_{\mathfrak{E}_1})\overset{\alpha',\gamma_*}{=\!=}A'_*\hat{\Theta}_2A+(I_{\mathfrak{E}_{*1}}-A'_*A_*)\hat{\Theta}_1-\hat{\Theta}_1=A'_*(\hat{\Theta}_2A-A_*\hat{\Theta}_1)\overset{\alpha}{=\!=}0,$$

$$\hat{\Delta}_1(A'A+D\hat{\Theta}_1-I_{\mathfrak{E}_1})\overset{\beta',\delta}{=\!=}(B'\hat{\Theta}_2+C'\hat{\Delta}_2)A-(B'A_*+C'B)\hat{\Theta}_1-\hat{\Theta}_1=$$
$$=B'(\hat{\Theta}_2A-A_*\hat{\Theta}_1)+C'(\hat{\Delta}_2A-B\hat{\Theta}_1)-\hat{\Delta}_1\overset{\alpha,\beta}{=\!=}C'C\hat{\Delta}_1-\hat{\Delta}_1=0,$$

and therefore

$$(\gamma)\qquad A'A+D\hat{\Theta}_1=I_{\mathfrak{E}_1}.$$

By analogous reasons, we have

$$(\gamma')\qquad AA'+D'\hat{\Theta}_2=I_{\mathfrak{E}_2}.$$

Furthermore,

$$\hat{\Theta}_2(AD-D'A_*)\overset{\alpha',\gamma'_*}{=\!=}A_*\hat{\Theta}_1D-(I_{\mathfrak{E}_{*2}}-A_*A'_*)A_*=A_*(\hat{\Theta}_1D+A'_*A_*)-A_*\overset{\gamma_*}{=\!=}0,$$

$$\hat{\Delta}_2(AD-D'A_*)\overset{\beta,\delta'}{=\!=}(B\hat{\Theta}_1+C\hat{\Delta}_1)D+(BA'_*+CB')A_*=$$
$$=B(\hat{\Theta}_1D+A'_*A_*)+C(\hat{\Delta}_1D+B'A_*)\overset{\gamma_*,\delta}{=\!=}B-CC'B\overset{\varepsilon}{=\!=}0,$$

and therefore

$$(\eta)\qquad AD=D'A_*.$$

By analogous reasons,

$$(\eta')\qquad A'D'=DA_*.$$

Conversely, if the operators occurring in (σ) except B' satisfy conditions (α)—(η') except (β'), (δ), (δ'), then the operator B' defined by

$$B'=-C'(\hat{\Delta}_2D'+BA'_*)$$

will obviously satisfy conditions $B'\in\mathscr{I}(S_{*2},R_1)$ and (δ'); let us show that it also satisfies (β') and (δ). Indeed, we have

$$B'\hat{\Theta}_2+C'\hat{\Delta}_2=-C'\hat{\Delta}_2D'\hat{\Theta}_2-C'BA'_*\hat{\Theta}_2+C'\hat{\Delta}_2=$$
$$\overset{\alpha',\gamma'}{=\!=}-C'\hat{\Delta}_2(I_{\mathfrak{E}_2}-AA')-C'B\hat{\Theta}_1A'+C'\hat{\Delta}_2=$$
$$=C'(\hat{\Delta}_2A-B\hat{\Theta}_1)A'\overset{\beta}{=\!=}C'C\hat{\Delta}_1A'\overset{\varepsilon}{=\!=}\hat{\Delta}_1A',$$

$$B'A_*+C'B = -C'\hat{\Delta}_2 D'A_* - C'BA'_*A_* + C'B =$$
$$\overset{\gamma_*,\eta}{=\!=} -C'\hat{\Delta}_2 AD - C'B(I_{\mathfrak{S}_{*1}} - \hat{\Theta}_1 D) + C'B =$$
$$= -C'(\hat{\Delta}_2 A - B\hat{\Theta}_2)D \overset{\beta}{=\!=} -C(C'\hat{\Delta}_1)D \overset{\varepsilon}{=\!=} -\hat{\Delta}_1 D.$$

Thus we have:

Theorem 1. 3′. *The contractions T_1, T_2 are similar if and only if there exist operators A_*, A'_*, A, A', D, D', C, C' and B satisfying conditions (σ) and (α), (α'), (β), (γ_*), (γ'_*), (γ), (γ'), (η), (η').*

6. Consider the particular case that T_2 is an *isometry*. Then $\mathfrak{R}_2=\mathfrak{H}_2$, $\mathfrak{S}_2=\{0\}$; thus A and D (whose *ranges* are in $\mathfrak{S}_2$) as well as A', $\hat{\Theta}_2$ and $\hat{\Delta}_2$ (which are *defined* on $\mathfrak{S}_2$) are all zero operators. Hence conditions (α)—(η') occurring in Theorem 1. 3′ reduce to the following ones:

$(\alpha)_0 \quad A_*\hat{\Theta}_1=0,$ $\qquad (\beta)_0 \quad B\hat{\Theta}_1+C\hat{\Delta}_1 = 0,$

$(\gamma_*)_0 \quad A'_*A_*+\hat{\Theta}_1 D = I_{\mathfrak{S}_{*2}},$ $\qquad (\varepsilon)_0 \quad C$: boundedly invertible,

$(\gamma'_*)_0 \quad A_*A'_*=I_{\mathfrak{S}_{*2}},$ $\qquad (\eta')_0 \quad DA'_*=0.$

$(\gamma)_0 \quad D\hat{\Theta}_1=I_{\mathfrak{S}_1},$

Thus in particular the existence of $D\in\mathscr{I}(S_{*1}, S_1)$ satisfying $(\gamma)_0$ is a necessary condition for T_1 to be similar to *some* isometry. This condition turns out to be also sufficient.

To this effect first observe that by account of relation $D\in\mathscr{I}(S_{*1}, S_1)$ the null-space $\mathfrak{D}=\ker D$ is invariant for S_{*1}. As S_{*1} is a unilateral shift so is $S_{*1}|\mathfrak{D}$ (possibly of multiplicity 0). Consider now the isometry

$$T_2 = (S_{*1}|\mathfrak{D})\oplus R_1 \quad \text{on} \quad \mathfrak{H}_2 = \mathfrak{D}\oplus\mathfrak{R}_1.$$

Then, clearly

$$\mathfrak{S}_{*2}=\mathfrak{D}, \quad \mathfrak{R}_2=\mathfrak{R}_1, \quad S_{*2}=S_{*1}|\mathfrak{D}, \quad \text{and} \quad R_2=R_1.$$

Set $A_* = I_{\mathfrak{S}_{*1}}-\hat{\Theta}_1 D$; by virtue of condition $(\gamma)_0$ we have $DA_* = (I_{\mathfrak{S}_1}-D\hat{\Theta}_1)D = 0$, whence $A_*:\mathfrak{S}_{*1}\to\mathfrak{S}_{*2}(=\mathfrak{D})$. The intertwining properties of $\hat{\Theta}_1$ and D imply that $A_*\in\mathscr{I}(S_{*1}, S_{*2})$. Furthermore, set

$$A'_*=I_{\mathfrak{S}_{*1}}|\mathfrak{D}, \quad B = -\hat{\Delta}_1 D, \quad C=C'=I_{\mathfrak{R}_1}.$$

It is easy to show that all the intertwining properties hold, and so do conditions $(\alpha)_0$—$(\eta')_0$; indeed,

$(\alpha)_0$: $\quad A_*\hat{\Theta}_1 = (I_{\mathfrak{S}_{*1}}-\hat{\Theta}_1 D)\hat{\Theta}_1 = \hat{\Theta}_1-(\hat{\Theta}_1 D)\hat{\Theta}_1 = 0, \quad$ by $(\gamma)_0$,

(β): $\quad B\hat{\Theta}_1+C\hat{\Delta}_1 = -\hat{\Delta}_1 D\hat{\Theta}_1+\hat{\Delta}_1 = 0, \quad$ by $(\gamma)_0$,

$(\gamma_*)_0$: $\quad A'_*A_*+\hat{\Theta}_1 D = (I_{\mathfrak{S}_{*1}}-\hat{\Theta}_1 D)+\hat{\Theta}_1 D = I_{\mathfrak{S}_{*1}},$

$(\gamma'_*)_0$: $\quad A_*A'_* = (I_{\mathfrak{S}_{*1}}-\hat{\Theta}_1 D)|\mathfrak{D} = I_{\mathfrak{S}_{*1}}|\mathfrak{D}=I_{\mathfrak{S}_{*2}},$

$(\eta')_0$: $\quad DA'_*=D|\mathfrak{D}=0.$

So we have proved:

Theorem 1.4. *The contraction T_1 is similar to some isometry if and only if $\hat{\Theta}_1$ has a left-inverse $D \in \mathscr{I}(S_{*1}, S_1)$. The unitary part of this isometry must be equal to R_1 (up to unitary equivalence).*

Corollary. *T_1 is similar to some unilateral shift if and only if $T_1^{*n} \to 0$ $(n \to \infty)$ and $\hat{\Theta}_1$ has a left inverse $D \in \mathscr{I}(S_{*1}, S_1)$.*

Proof. Necessity of $T_1^{*n} \to 0$ follows from the same property of unilateral shifts. On the other hand, this condition is equivalent to $\mathfrak{R}_1 = \{0\}$; *cf.* [5], Chapter II, Theorem 1.2 and formulas (2.1), (2.7). The isometry to which T_1 is similar by virtue of Theorem 1.4 must therefore have $\mathfrak{R}_2 = \{0\}$, i.e. be a unilateral shift.

7. If T_2 is *unitary*, we not only have $\mathfrak{S}_2 = \{0\}$, but $\mathfrak{S}_{*2} = \{0\}$ as well, so the operators A_*, A'_* are also zero, and the set of conditions $(\alpha)_0$—$(\eta')_0$ reduces to the following:

$$(\beta)_{00} \quad B\hat{\Theta}_1 + C\hat{\Delta}_1 = 0, \quad (\gamma_*)_{00} \quad \hat{\Theta}_1 D = I_{\mathfrak{S}_{*1}}, \quad (\gamma)_{00} \quad D\hat{\Theta}_1 = I_{\mathfrak{S}_1},$$

with C boundedly invertible. Thus a necessary condition for T_1 to be similar to some unitary operator is that $\hat{\Theta}_1$ be boundedly invertible. This condition is also sufficient. For, if we choose for T_2 any unitary operator U unitarily equivalent to R_1 and for C any unitary operator satisfying $UC = CR_1$, then the operators

$$D = \hat{\Theta}_1^{-1} \quad \text{and} \quad B = -C\hat{\Delta}_1 \hat{\Theta}_1^{-1}$$

will obviously satisfy the conditions above as well as the intertwining conditions (σ).

So we have

Theorem 1.5. *The contraction T_1 is similar to some unitary operator if and only if the operator $\hat{\Theta}_1$ is boundedly invertible. This unitary operator must then be equal to R_1 (up to unitary equivalence).*

See [5], Sec. IX. 1 for another proof.

2. Completely non-unitary contractions

1. For c.n.u. contractions we shall use their functional model. All Hilbert spaces to be considered are separable.

For a Hilbert space $\mathfrak{E}$, $L^2(\mathfrak{E})$ will denote the Hilbert space of $\mathfrak{E}$-vector valued functions $u = u(z)$ on the unit circle $(z = e^{it})$, which are (strongly) measurable and norm-square integrable with respect to normed Lebesgue measure, i.e. with

$$\|u\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |u(z)|^2 \, dt \right)^{1/2},$$

where $|\cdot|$ denotes vector norm in $\mathfrak{E}$. Then $H^2(\mathfrak{E})$ is the Hardy subspace of $L^2(\mathfrak{E})$.

We shall be also considering functions $\Phi=\Phi(z)$, whose values are operators from a Hilbert space $\mathfrak{E}$ into a Hilbert space $\mathfrak{F}$; we require that these operator-valued functions be (strongly) measurable and essentially bounded, i.e. with

$$\operatorname{ess\,sup} |\Phi(z)|<\infty;$$

here $|\cdot|$ denotes the norm of operator from $\mathfrak{E}$ into $\mathfrak{F}$. Multiplication on $L^2(\mathfrak{E})$ by such a bounded measurable function Φ is an operator from $L^2(\mathfrak{E})$ into $L^2(\mathfrak{F})$, which we denote by the same letter Φ; thus

$$(\Phi u)(z)=\Phi(z)u(z) \quad (u\in L^2(\mathfrak{E})).$$

Note that the norm $\|\Phi\|$ of this operator equals the essential supremum of $|\Phi(z)|$. In particular, the operator Φ is a contraction if and only if the function Φ is "contractive", i.e. if its values are contractions $\Phi(z):\mathfrak{E}\to\mathfrak{F}$ a.e. on the unit circle.

A bounded measurable function Φ is *analytic* if the corresponding operator Φ maps the subspace $H^2(\mathfrak{E})$ of $L^2(\mathfrak{E})$ into the subspace $H^2(\mathfrak{F})$ of $L^2(\mathfrak{F})$, or equivalently, if its values $\Phi(z)$ are the radial (strong) limits, a.e. on the unit circle, of a bounded holomorphic function $\Phi(\lambda)$ in the open unit disc, $|\lambda|<1$.

Let Θ be a contractive analytic function with values operators $\Theta(z):\mathfrak{E}\to\mathfrak{E}_*$, and which is, moreover, "pure" in the sense that it also satisfies

$$|\Theta(0)a|<|a| \quad \text{for all} \quad a\in\mathfrak{E}, \quad a\neq 0.$$

We associate with Θ the function

$$\Delta(z) = [I_{\mathfrak{E}}-\Theta(z)^*\Theta(z)]^{1/2},$$

which is also measurable and whose values are selfadjoint operators on $\mathfrak{E}$, bounded by 0 and 1. We form the Hilbert space

$$\mathfrak{K} = H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}$$

(where the closure is in the metric of $L^2(\mathfrak{E})$) and its subspace

$$\mathfrak{H} = \left[H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}\right]\ominus\{\Theta u\oplus\Delta u:\ u\in H^2(\mathfrak{E})\},$$

and define on $\mathfrak{H}$ the operator $S(\Theta)$ by

$$S(\Theta)(u\oplus v) = P_{\mathfrak{H}}(\chi u\oplus\chi v),$$

where $\chi(z)\equiv z$ and $P_{\mathfrak{H}}$ denotes orthogonal projection of $\mathfrak{K}$ onto its subspace $\mathfrak{H}$.

This operator $S(\Theta)$ is a c.n.u. contraction, and moreover, one obtains in this way *all* c.n.u. contractions T, up to unitary equivalence. For T given, one has indeed to choose for $\Theta(\lambda)$ the "characteristic" function of T. See [5], Chapter VI.

The operator V defined on the space $\mathfrak{K}$ by

$$V(u\oplus v) = \chi u\oplus\chi v$$

turns out to be the minimal isometric dilation of $T=S(\Theta)$, and in the Wold decomposition of $\mathfrak{K}$ for V we have

$$\mathfrak{S}_*=H^2(\mathfrak{E}_*) \quad \text{and} \quad \mathfrak{R}=\overline{\Delta L^2(\mathfrak{E})}$$

(with the natural embeddings in $\mathfrak{K}$ as $H^2(\mathfrak{E}_*)\oplus\{0\}$ and $\{0\}\oplus\overline{\Delta L^2(\mathfrak{E})}$). The corresponding parts S_* and R of V both are multiplication by χ. On the other hand we have

$$\mathfrak{S} = \mathfrak{K}\ominus\mathfrak{H} = \{\Theta u\oplus\Delta u : u\in H^2(\mathfrak{E})\}.$$

As $u \to \Theta u\oplus\Delta u$ is a unitary map of $H^2(\mathfrak{E})$ onto $\mathfrak{S}$, which commutes with multiplication by χ, it is justified to identify $\mathfrak{S}$ with $H^2(\mathfrak{E})$; S will then be represented by multiplication by χ on $H^2(\mathfrak{E})$. The projection operators $\hat{\Theta}$ and $\hat{\Delta}$, from $\mathfrak{S}$ into $\mathfrak{S}_*$ and $\mathfrak{R}$, will be represented by the restrictions to $H^2(\mathfrak{E})$ of the operators Θ and Δ, respectively.

We shall use the fundamental fact that if $\mathfrak{E}$ and $\mathfrak{E}'$ are Hilbert spaces, and if Ω and Ω' are bounded measurable functions with values operators

$$\Omega(z):\mathfrak{E}\to\mathfrak{E}, \quad \Omega'(z):\mathfrak{E}'\to\mathfrak{E}',$$

then those operators

a) Φ: $H^2(\mathfrak{E})\to H^2(\mathfrak{E}')$,

b) Φ: $H^2(\mathfrak{E})\to\overline{\Omega' L^2(\mathfrak{E}')}$,

c) Φ: $\overline{\Omega L^2(\mathfrak{E})}\to\overline{\Omega' L^2(\mathfrak{E}')}$

which commute with multiplication by χ can be represented as multiplication (on the left) by an operator valued, bounded function $\Phi(\cdot)$ which is

a) analytic, with values $\Phi(z)$: $\mathfrak{E}\to\mathfrak{E}'$ a.e.,

b) measurable, with values $\Phi(z)$: $\mathfrak{E}\to\overline{\Omega'(z)\mathfrak{E}'}$ a.e.,

c) measurable, with values $\Phi(z)$: $\overline{\Omega(z)\mathfrak{E}}\to\overline{\Omega'(z)\mathfrak{E}'}$ a.e.

Here, in case c), "measurability" means that there exists a measurable function Ψ with values $\Psi(z):\mathfrak{E}\to\mathfrak{E}'$ such that

$$\Phi(z) = \Psi(z)|\overline{\Omega(z)\mathfrak{E}} \quad \text{a. e.}$$

For the case a) the above fact is proved e.g. in [5], Sec. V. 3; the cases b) and c) can be dealt with in an analogous manner.

2. Consider now *two* c.n.u. contractions, or rather their functional models, say

$$T_1=S(\Theta_1) \quad \text{and} \quad T_2=S(\Theta_2),$$

where Θ_k are purely contractive analytic functions with values operators $\Theta_k(z):\mathfrak{E}_k\to\mathfrak{E}_{*k}$ $(k=1, 2)$. Then

$$\mathfrak{K}_k = H^2(\mathfrak{E}_{*k})\oplus\overline{\Delta_k L^2(\mathfrak{E}_k)} \quad (k=1, 2)$$

are the corresponding dilation spaces; the elements of $\mathfrak{K}_k$ can also be thought of as column vectors $\begin{bmatrix} u \\ v \end{bmatrix}$.

For these operators, Lemmas 1. 1 and 1. 2 appear in the following form:

Lemma 2. 1. *The general form of an operator $Y\in\mathscr{I}^+(T_1, T_2)$ is multiplication (on $\mathfrak{K}_1$) by a matrix function*

$$Y(z)=\begin{bmatrix} A_*(z) & 0 \\ B(z) & C(z)\end{bmatrix} \tag{2. 1}$$

where A_* is a bounded analytic function and B, C are bounded measurable functions with values operators

$$A_*(z)\colon \mathfrak{E}_{*1}\to\mathfrak{E}_{*2},\quad B(z)\colon \mathfrak{E}_{*1}\to\overline{\Delta_2(z)\mathfrak{E}_1},\quad C(z)\colon \overline{\Delta_1(z)\mathfrak{E}_1}\to\overline{\Delta_2(z)\mathfrak{E}_2}, \tag{2. 2}$$

a.e. on the unit circle, satisfying the conditions

$$A_*\Theta_1=\Theta_2 A,\quad B\Theta_1+C\Delta_1 = \Delta_2 A, \tag{2. 3}$$

where A is some bounded analytic function with values operators

$$A(z)\colon\ \mathfrak{E}_1\to\mathfrak{E}_2 \quad \text{a.e.} \tag{2. 4}$$

Lemma 2. 2. *The general form of an operator $Y\in\mathscr{I}^+(T_1, T_2)$ satisfying $\pi_{12}(Y)=0$ is multiplication (on $\mathfrak{K}_1$) by a matrix function*

$$\begin{bmatrix} \Theta_2(z)D(z) & 0 \\ \Delta_2(z)D(z) & 0\end{bmatrix},$$

where D is a bounded analytic function, with values operators

$$D(z)\colon\ \mathfrak{E}_{*1}\to\mathfrak{E}_2 \quad \text{a.e.} \tag{2. 5}$$

3. Let us consider besides the functions $\Delta_k(z)$ their duals

$$\Delta_{*k}(z) = [I_{\mathfrak{E}_{*k}}-\Theta_k(z)\Theta_k(z)^*]^{1/2} \quad (k=1, 2).$$

Then $\Theta_k\Delta_k=\Delta_{*k}\Theta_k$. Suppose A_*, A, B, C are functions satisfying the conditions of Lemma 2. 1 and derive from them the function

$$E(z) = [B(z)\Delta_{*1}(z)-C(z)\Theta_1(z)^*]\,|\,\overline{\Delta_{*1}(z)\mathfrak{E}_{*1}}.$$

Clearly, E is a bounded measurable function such that

$$E(z)\colon \overline{\Delta_{*1}(z)\mathfrak{E}_{*1}} \to \overline{\Delta_2(z)\mathfrak{E}_2} \quad \text{a. e.} \tag{2. 6}$$

Then, using (2. 3) we get

$$E\Theta_1\Delta_1 = (B\Delta_{*1}\Theta_1 - C\Theta_1^*\Theta_1)\Delta_1 = (B\Theta_1\Delta_1 - C + C\Delta_1^2)\Delta_1 = (\Delta_2 A\Delta_1 - C)\Delta_1$$

and therefore

$$C(z) = [-E(z)\Theta_1(z) + \Delta_2(z)A(z)\Delta_1(z)] \,|\, \overline{\Delta_1(z)\mathfrak{E}_1}. \tag{2. 7}$$

Furthermore, we have

$$E\Delta_{*1} = B\Delta_{*1}^2 - C\Theta_1^*\Delta_{*1} = B - B\Theta_1\Theta_1^* - C\Delta_1\Theta_1^*;$$

and hence by (2. 3):

$$B = E\Delta_{*1} + \Delta_2 A\Theta_1^*. \tag{2. 8}$$

Conversely, for an arbitrary bounded measurable function E with values operators as in (2. 6), the functions B and C generated by (2. 7) and (2. 8) will satisfy conditions (2. 2) and (2. 3). Indeed, we have in particular

$$\begin{aligned} B\Theta_1 + C\Delta_1 &= [E\Delta_{*1} + \Delta_2 A\Theta_1^*]\Theta_1 + [-E\Theta_1 + \Delta_2 A\Delta_1]\Delta_1 = \\ &= E[\Delta_{*1}\Theta_1 - \Theta_1\Delta_1] + \Delta_2 A = \Delta_2 A. \end{aligned}$$

Thus we can give Lemma 2. 1 the following alternative form:

Lemma 2. 1′. *The general form of an operator $Y \in \mathscr{I}^+(T_1, T_2)$ is multiplication by a matrix function* (2. 1), *where A_* is as in Lemma* 2. 1,[1)] *while B and C derive by means of formulas* (2. 7) *and* (2. 8) *from some bounded measurable function E with values operators*

$$E(z)\colon \overline{\Delta_{*1}(z)\mathfrak{E}_{*1}} \to \overline{\Delta_2(z)\mathfrak{E}_2}.$$

4. The similarity theorems 1. 3 and 1. 3′ can be formulated for operators $T_k = S(\Theta_k)$ $(k=1, 2)$ as follows:

Theorem 2. 3. *The operators $S(\Theta_k)$ $(k=1, 2)$ are similar if and only if there exist bounded analytic functions A_*, A'_*, A, A', D, D' and bounded measurable functions B, B', C, C' with values operators*

$$(\sigma_1)\begin{cases} A_*(z)\colon \mathfrak{E}_{*1} \to \mathfrak{E}_{*2}, & A(z)\colon \mathfrak{E}_1 \to \mathfrak{E}_2, \quad D(z)\colon \mathfrak{E}_{*1} \to \mathfrak{E}_1, \\ A'_*(z)\colon \mathfrak{E}_{*2} \to \mathfrak{E}_{*1}, & A'(z)\colon \mathfrak{E}_2 \to \mathfrak{E}_1, \quad D'(z)\colon \mathfrak{E}_{*2} \to \mathfrak{E}_2, \end{cases}$$

[1)] I. e. bounded analytic with values operators $\mathfrak{E}_{*1} \to \mathfrak{E}_{*2}$, satisfying the relation $A_*\Theta_1 = \Theta_2 A$, where A is some bounded analytic function with values operators $\mathfrak{E}_1 \to \mathfrak{E}_2$.

$$(\sigma_2)\begin{cases} B(z)\colon \mathfrak{E}_{*1}\to\overline{\Delta_2(z)\mathfrak{E}_2}, & C(z)\colon \overline{\Delta_2(z)\mathfrak{E}_2}\to\overline{\Delta_1(z)\mathfrak{E}_1},\\ B'(z)\colon \mathfrak{E}_{*2}\to\overline{\Delta_1(z)\mathfrak{E}_1}, & C'(z)\colon \overline{\Delta_1(z)\mathfrak{E}_1}\to\overline{\Delta_2(z)\mathfrak{E}_2},\end{cases}$$

satisfying a.e. the conditions

(α) $A_*(z)\Theta_1(z)=\Theta_2(z)A(z)$, (β) $B(z)\Theta_1(z)+C(z)\Delta_1(z)=\Delta_2(z)A(z)$,

(α') $A'_*(z)\Theta_2(z)=\Theta_1(z)A'(z)$, (β') $B'(z)\Theta_2(z)+C'(z)\Delta_2(z)=\Delta_1(z)A'(z)$,

(γ_*) $A'_*(z)A_*(z)+\Theta_1(z)D(z)=I_{\mathfrak{E}_{*1}}$, (δ) $B'(z)A_*(z)+C'(z)B(z)=-\Delta_1(z)D(z)$,

(γ'_*) $A_*(z)A'_*(z)+\Theta_2(z)D'(z)=I_{\mathfrak{E}_{*2}}$, (δ') $B(z)A'_*(z)+C(z)B'(z)=-\Delta_2(z)D'(z)$,

(ε) $C'(z)=C(z)^{-1}$.

Theorem 2.3′. *The operators $S(\Theta_k)$ $(k=1,2)$ are similar if and only if there exist bounded analytic functions A_*, A'_*, A, A', D, D', and bounded measurable functions B, C and C' satisfying conditions (σ), (α), (α'), (β), (γ_*), (γ'_*) of Theorem 2.3 and conditions*

(γ) $A'(z)A(z)+D(z)\Theta_1(z)=I_{\mathfrak{E}_1}$, (η) $A(z)D(z)=D'(z)A_*(z)$,

(γ') $A(z)A'(z)+D'(z)\Theta_2(z)=I_{\mathfrak{E}_2}$, (η') $A'(z)D'(z)=D(z)A'_*(z)$.

Corollary 1. *The equation*

$$\dim\overline{\Delta_1(z)\mathfrak{E}_1}=\dim\overline{\Delta_2(z)\mathfrak{E}_2}\quad \text{a. e.} \tag{2.9}$$

is a necessary condition for $S(\Theta_1)$ and $S(\Theta_2)$ to be similar.

Proof. Immediate from the invertibility of $C(z)$ a.e.

We shall return in Sec. 3 to the question how (2. 9) can replace in some cases the conditions on B, C, C' in Theorem 2. 3′.

Consider now the case that $\Theta_k(z)$ $(k=1,2)$ are *inner* functions (i.e. with values isometries a.e. on the unit circle). Then $\Delta_k(z)=0$ $(k=1,2)$ a.e., and hence, by (σ_2), the values of B, B', C, C' are operators with range $\{0\}$ so that conditions (β), (β'), (δ), (δ'), (ε) of Theorem 2. 3 become trivial. Thus we have:

Corollary 2. *If $\Theta_k(z)$ $(k=1,2)$ are inner functions then conditions (σ_1), (α) (α'), (γ_*), (γ'_*) are necessary and sufficient for $S(\Theta_1)$ and $S(\Theta_2)$ to be similar.*

Taking $D=0$, $D'=0$ we get a sufficient condition:

Corollary 3. *If $\Theta_k(z)$ $(k=1,2)$ are inner functions then for the similarity of $S(\Theta_1)$ and $S(\Theta_2)$ it is sufficient that there exist bounded analytic functions $A_*(z)$, $A(z)$ with bounded inverses $A_*(z)^{-1}$, $A(z)^{-1}$ such that*

$$A_*(z)\Theta_1(z)=\Theta_2(z)A(z)\quad \text{a. e.}$$

This sufficiency condition was obtained in a direct manner in Moore—Nordgren [4].

5. Theorems 1. 4 and 1. 5 can also be given a functional form, and here one need not restrict himself to c.n.u. operators. Indeed every contraction T is the direct sum of a unitary operator and of a c.n.u. contraction T_1. Clearly T is similar to an isometry or to a unitary operator if and only if so does T_1. As T and T_1 have the same characteristic function Θ we deduce from the theorems above:

Theorem 2. 4. *A contraction T is similar to some isometry or to some unitary operator if and only if its characteristic function Θ has a bounded analytic left-inverse, or inverse, respectively.*

(For the unitary case see also [5], Sec. IX. 1.)

3. An equidimensionality criterion for similarity

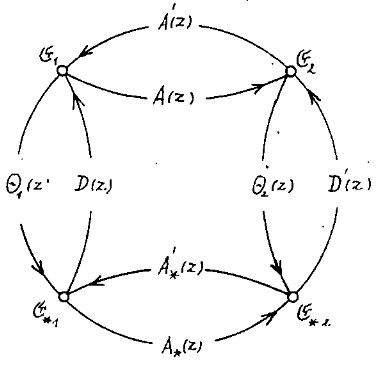
1. The following theorem differs from Theorems 2. 3 and 2. 3′ in that it only involves the analytic functions $A_*, \ldots, D'$, plus the equidimensionality condition (2. 9). More precisely, we prove

Theorem 3. 1. *Suppose $\Theta_k(z)$ $(k=1, 2)$ are purely contractive analytic functions, with values operators $\mathfrak{E}_k \to \mathfrak{E}_{*k}$, and also suppose that the values of the function $\Delta_1(z)$ are compact operators a.e. on the unit circle. Then $S(\Theta_1)$ is similar to $S(\Theta_2)$ if and only if*

$$(*) \qquad \dim \overline{\Delta_1(z)\mathfrak{E}_1} = \dim \overline{\Delta_2(z)\mathfrak{E}_2} \quad a.\ e.$$

and if there exist bounded analytic functions $A_*(z), \ldots, D'(z)$ satisfying conditions (σ_1), (α), (α'), (γ_*), (γ'_*), (γ), (γ'), (η), (η') of Theorems 2. 3 and 2. 3′.

Observe that these conditions can be expressed by the following properties of the diagram: 1) it is commutative along *open* two-step paths (e.g. $AD=D'A_*$); 2) products corresponding to adjacent *closed* two-step paths add to identity (e.g. $A'A+D\Theta_1 = I_{\mathfrak{E}_1}$).



Proof. Necessity follows from Theorem 2. 3′ and its Corollary 1. So we have to prove sufficiency, that is, on account of Theorem 2. 3′, the existence of functions $B(z)$, $C(z)$, $C'(z)$ satisfying conditions (σ_2), (β), (ε). This will be done as follows.

a) First we introduce the spectral family $\{E_x(z)\}_{0\leqq x\leqq 1}$ of the selfadjoint operator $\Delta_1(z)$, normed by the conditions $E_0(z)=0$, $E_1(z)=I$, and, say, continuity from the right for $0<x<1$. As $E_x(z)$ is the limit of a sequence of polynomials of $\Delta_1(z)$, we conclude that $E_x(z)$ is, for any fixed x, a measurable function of $z=e^{it}$ along with $\Theta(z)$ and $\Delta(z)$.

Note that if $a\in[I_{\mathfrak{E}_1}-E_x(z)]\mathfrak{E}_1$ for some x and z ($0\leqq x\leqq 1$, $z=e^{it}$), then

$$(3.1)\qquad |\Delta_1(z)a|\geqq x|a|,\quad |\Theta_1(z)a|^2 = |a|^2-|\Delta_1(z)a|^2 \leqq (1-x^2)|a|^2$$

(norms in $\mathfrak{E}_1$ and $\mathfrak{E}_{*1}$).

Denote by M the least common upper bound of the functions

$$|A_*(z)|, \ldots, |D'(z)|.$$

For vectors a of the type considered above we have then

$$|a-A'(z)A(z)a| \overset{\gamma}{=} |D(z)\Theta_1(z)a| \leqq M|\Theta_1(z)a| \leqq M(1-x^2)^{1/2}|a|.$$

As we also have

$$|A'(z)A(z)a|\leqq M|A(z)a|$$

we deduce: $|a| \leqq M(1-x^2)^{1/2}|a|+M|A(z)a|$, and hence

$$(3.2)\qquad |A(z)a| \geqq [M^{-1}-(1-x^2)^{1/2}]|a| \geqq \frac{3}{4M}|a|$$

for x close enough to 1. Moreover, we have

$$|\Delta_2(z)A(z)a|^2 = |A(z)a|^2-|\Theta_2(z)A(z)a|^2 =$$

$$\overset{\alpha}{=} |A(z)a|^2-|A_*(z)\Theta_1(z)a|^2 \geqq |A(z)a|^2-M^2|\Theta_1(z)a|^2;$$

using (3. 1) and (3. 2) we get

$$(3.3)\qquad |\Delta_2(z)A(z)a| \geqq \left[\left(\frac{3}{4M}\right)^2-M^2(1-x^2)\right]|a|^2 \geqq \left(\frac{1}{2M}\right)^2|a|^2$$

for x close enough to 1 ($0<x<1$). Let us fix such a value of x, say ξ, and denote

$$E_\xi^-(z) = I_{\mathfrak{E}_1}-E_\xi(z).$$

On account of (3. 3) we have then for $a\in E_\xi^-(z)\mathfrak{E}_1$

$$(3.4)\qquad |\Delta_2(z)A(z)a| \geqq (2M)^{-1}|a| \geqq (2M)^{-1}|\Delta_1(z)a|.$$

On the other hand, (3. 1) implies, for such a,

(3. 5) $$|\Delta_2(z)A(z)a| \leqq |A(z)a| \leqq M|a| \leqq (M/\xi)|\Delta_1(z)a|.$$

b) Now set

(3. 6) $$F(z) = \int_{\xi}^{1} \frac{1}{x} d_x E_x(z) \quad \text{and} \quad G(z) = \Delta_2(z)A(z)F(z);$$

these are bounded, measurable functions, and as

(3. 7) $$F(z)a \in E_\xi^- \mathfrak{E}_1 \quad \text{and} \quad \Delta_1(z)F(z)a = a \quad \text{for} \quad a \in E_\xi^- \mathfrak{E}_1$$

we deduce from (3. 4) and (3. 5) that

(3. 8) $$(2M)^{-1}|a| \leqq |G(z)a| \leqq (M/\xi)|a| \quad \text{for} \quad a \in E_\xi^- \mathfrak{E}_1.$$

Because $\Delta_1(z)$ is a compact operator for a.e. z, its spectral projection $E_\xi^-(z)$ is of finite rank, a.e. By virtue of (3. 8), $G(z)$ maps $E_\xi^-(z)\mathfrak{E}_1$, for a.e. fixed value of z, bicontinuously onto the space

$$G(z)E_\xi^-(z)\mathfrak{E}_1 \quad (\subset \Delta_2(z)\mathfrak{E}_2),$$

which is therefore of the same dimension as $E^-(z)\mathfrak{E}_1$. From the hypothesis $(*)$ it then follows that the spaces

$$\mathfrak{M}_1(z) = \overline{\Delta_1(z)\mathfrak{E}_1} \ominus E_\xi^- \mathfrak{E}_1 \quad \text{and} \quad \mathfrak{M}_2(z) = \overline{\Delta_2(z)\mathfrak{E}_2} \ominus G(z)E_\xi^-(z)\mathfrak{E}_1$$

are also equidimensional. Thus

$$\dim \mathfrak{M}_1(z) = \dim \mathfrak{M}_2(z) = d(z) \quad \text{a.e.},$$

where $d(z)$ is a measurable function with the possible values $0, 1, \ldots, \infty (= \aleph_0)$. Note that

$$\overline{\Delta_1(z)\mathfrak{E}_1} = (I_{\mathfrak{E}_1} - E_{+0}(z))\mathfrak{E}_1, \quad \text{and hence} \quad \mathfrak{M}_1(z) = (E_\xi(z) - E_{+0}(z))\mathfrak{E}_1.$$

By an appropriate orthogonalization procedure (commonly used in reduction theory) we construct sequences $\{\varphi_k\}_1^\infty$, $\{\psi_k\}_1^\infty$ of $\mathfrak{E}_1$- and $\mathfrak{E}_2$-vector valued measurable functions such that if σ_n denotes, for $n = 0, 1, \ldots, \infty$, the set of points z on the unit circle where

$$d(z) = n,$$

then for every fixed $z \in \sigma_n$ the values

$$\varphi_1(z), \ldots, \varphi_n(z) \quad \text{and} \quad \psi_1(z), \ldots, \psi_n(z)$$

form orthonormal bases of $\mathfrak{M}_1(z)$ and $\mathfrak{M}_2(z)$, respectively. Then it is easy to define a measurable function with values *unitary* operators

$$U(z)\colon\ \mathfrak{M}_1(z)\to\mathfrak{M}_2(z),$$

notably we set

$$U(z)\varphi_k(z)=\psi_k(z)\quad\text{for}\quad z\in\sigma_n\quad\text{and}\quad k=1,\dots,n,$$

and extend linearly. (On σ_0 we can set e.g. $U(z)=0$.)

Now consider the function

$$C(x)=\left[G(z)E_\xi^-(z)+U(z)\left(E_\xi(z)-E_{+0}(z)\right)\right]\Big|\overline{\Delta_1(z)\mathfrak{E}_1}\,; \tag{3.9}$$

this is also measurable and its values are operators

$$C(z)\colon\ \overline{\Delta_1(z)\mathfrak{E}_1}\to\overline{\Delta_1(z)\mathfrak{E}_2}\quad\text{(onto)} \tag{3.10}$$

satisfying, by virtue of (3. 8), the inequalities

$$M_1|a|\leqq|C(z)a|\leqq M_2|a|\quad\text{for}\quad a\in\overline{\Delta_1(z)\mathfrak{E}_1} \tag{3.11}$$

with the constants

$$M_1=\min\{1,(2M)^{-1}\},\quad M_2=\max\{1, M/\xi\}.$$

Hence the function

$$C'(z)=C(z)^{-1}$$

has sense, is measurable and bounded,

$$|C'(z)|\leqq 1/M_1.$$

c) Consider now the function

$$H(z)=\Delta_2(z)A(z)-C(z)\Delta_1(z). \tag{3.12}$$

As

$$C(z)\Delta_1(z)E_\xi^-(z)=C(z)E_\xi^-(z)\Delta_1(z)=G(z)E_\xi^-(z)\Delta_1(z)=$$
$$=\Delta_2(z)A(z)F(z)\Delta_1(z)E_\xi^-(z)=\Delta_2(z)A(z)E_\xi^-(z)$$

by virtue (3. 9), (3. 6) and (3. 7), we have $H(z)E_\xi^-(z)=0$. Therefore and from (3. 9) and (3. 6) we have

$$H(z)=H(z)E_\xi(z)=\Delta_2(z)A(z)E_\xi(z)-U(z)\Delta_1(z)\left(E_\xi(z)-E_{+0}(z)\right).$$

Hence,

$$|H(z)a|\leqq(M+1)|E_\xi(z)a|\quad\text{for}\quad a\in\mathfrak{E}_1.$$

On the other hand,

$$|\Theta_1(z)a|^2 = |a|^2 - |\Delta_1(z)a|^2 = \int_0^1 (1-x^2)\, d_x|E_x(z)a|^2 \geqq$$

$$\geqq (1-\xi^2)|E_\xi(z)a|^2 \quad \text{for} \quad a \in \mathfrak{E}_1;$$

combination of the two results gives

$$|H(z)a| \leqq N \cdot |\Theta_1(z)a| \quad \text{for} \quad a \in \mathfrak{E}_1$$

with the constant $N = (M+1)(1-\xi^2)^{-1/2}$.

This shows that, for a.e. fixed z, the operator $B_0(z)$ defined on $\Theta_1(z)\mathfrak{E}_1$ by

$$B_0(z)\Theta_1(z)a = H(z)a \qquad (a \in \mathfrak{E}_1) \tag{3. 13}$$

is (linear and) bounded by N; its definition extends by continuity to the closure of $\Theta_1(z)\mathfrak{E}_1$. Denote by $P(z)$ the orthogonal projection of $\mathfrak{E}_{*1}$ onto its subspace $\overline{\Theta_1(z)\mathfrak{E}_1}$ and set

$$B(z) = B_0(z)P(z). \tag{3. 14}$$

On account of (3. 12) and (3. 10) the range of $H(z)$ is contained in $\overline{\Delta_2(z)\mathfrak{E}_2}$. From these results we conclude:

$$B(z)\colon \mathfrak{E}_{*1} \to \overline{\Delta_2(z)\mathfrak{E}_2}, \quad |B(z)| \leqq N,$$

and

$$B(z)\Theta_1(z) = B_0(z)P(z)\Theta_1(z) = B_0(z)\Theta_1(z) = H(z), \tag{3. 15}$$

i.e. the function $B(z)$ is bounded and satisfies conditions (σ_2) and (β) of Theorem 2. 3. It remains to prove that it is also measurable.

To this effect first note that, by its definition (3. 12), $H(z)$ is measurable. From (3. 15) it follows therefore that $B(z)u(z)$ is measurable for every function $u \in \Theta_1 L^2(\mathfrak{E}_1)$, and hence for every function $u \in \overline{\Theta_1 L^2(\mathfrak{E}_1)}$ also. Next note that since $\Theta_1(z)$ is a measurable function, so is $P(z)$; and hence $w(z) = P(z)v(z)$ is measurable for every $\mathfrak{E}_1$-vector valued measurable function v, in particular for every function $v \in L^2(\mathfrak{E}_1)$. Now in this case it is obvious that w is the orthogonal projection of v onto the subspace $\overline{\Theta_1 L^2(\mathfrak{E}_1)}$ of $L^2(\mathfrak{E}_1)$. Since we have, moreover,

$$B(z)v(z) = B(z)w(z)$$

on account of (3. 14), we conclude that $B(z)v(z)$ is measurable for every $v \in L^2(\mathfrak{E}_1)$, thus $B(z)$ itself is measurable.

This completes the proof of Theorem 3. 1. Observe that the compacity assumption on $\Delta_1(z)$ can be weakened: all we have used is that, for a certain $\xi < 1$, the spectral projection $E_\xi^-(z)$ is of finite rank, a.e.

2. If both $\Theta_1(z)$ and $\Theta_2(z)$ are *scalar* valued then so are all functions occurring in Theorems 2. 3, 2. 3′, and 3. 1, and therefore commute. From conditions (α), (γ) it follows in this case

$$A_*\Theta_1 A' = \Theta_2 AA' = \Theta_2 A'A = \Theta_2(I-D\Theta_1), \quad \Theta_2 = (A_*A'+\Theta_2 D)\Theta_1;$$

and from (α) and (γ_*') we get on a similar way

$$\Theta_1 = (A_*'A+D'\Theta_1)\Theta_2.$$

Thus both functions

(3. 16) $\qquad \Theta_1(z)/\Theta_2(z)$ and $\Theta_2(z)/\Theta_1(z)$ belong to H^∞.

Conversely, if (3. 16) holds then conditions (α)—(η') are fulfilled e.g. by the functions

$$A_*=\Theta_2/\Theta_1,\ A=1,\ D=0;\quad A_*'=\Theta_1/\Theta_2,\ A'=1,\ D'=0.$$

Clearly, $\dim \Delta_k(z)\mathfrak{E}_k$ is 0 or 1 according as $|\Theta_k(z)|$ is <1 or $=1$. Thus we obtain from Theorem 3. 1:

Corollary 1. *Let $\Theta_1(z)$, $\Theta_2(z)$ be scalar valued contractive analytic functions. Then the corresponding operators $S(\Theta_1)$, $S(\Theta_2)$ are similar if and only if*

(i) *$\Theta_1(z)/\Theta_2(z)$ and $\Theta_2(z)/\Theta_1(z)$ belong to H^∞,*

(ii) *the sets $\{z:|\Theta_k(z)|=1\}$ $(k=1, 2)$ coincide up to subsets of zero measure.*

This result was obtained earlier by KRIETE [3].

3. We shall now consider an $N\times N$ matrix valued, purely contractive analytic function

$$\Theta(z)=[\vartheta_{ik}(z)] \qquad (i, k=1, 2, \dots, N).$$

Let

$$\Omega(z)=[\omega_{ik}(z)] \qquad (i, k=1, 2, \dots, N)$$

denote the algebraic adjoint matrix; then

(3. 17) $\qquad \Omega(z)\Theta(z)=\Theta(z)\Omega(z)=d(z)I_N, \quad$ where $\quad d(z)=\det\Theta(z).$

If Θ is *inner*, it is known that the operator $S(\Theta)$ is *quasisimilar* to an operator $S(\vartheta)$ generated by a scalar valued inner function $\vartheta(z)$ if and only if the functions $\omega_{ik}(z)$ have no non-constant inner common divisor, and in this case we necessarily have $\vartheta=d$. (See [3], Sec. IX. 2, and [6].)

Still in the case of inner Θ, a necessary condition for $S(\Theta)$ to be *similar* to $S(d)$ was proved in [8], Sec. 8 (see in particular the last row on p. 17). In an equivalent form, this condition reads as follows:

$$\text{(c)}\quad \begin{cases} \text{There exist functions } u_i, v_i, w \in H^\infty \quad (i=1, \dots, N) \text{ such that} \\ \sum_{i,j=1}^{N} u_i \omega_{ij} v_j + dw = 1. \end{cases}$$

Necessity of condition (c) easily follows, even for not necessarily inner Θ, from Theorem 2. 3 when applied to $\Theta_1 = \Theta$ and $\Theta_2 = d$. Indeed, conditions (α), (γ'_*) and equation (3. 17) give:

$$A_* \Theta = dA, \quad A_* A'_* + dD = 1, \quad \Omega\Theta = \Theta\Omega = d \cdot I_N;$$

hence

$$(dA)\Omega A'_* = (A_* \Theta)\Omega A'_* = A_* dA'_* = d(1 - dD')$$

and dividing by d,

$$A\Omega A'_* = 1 - dD'. \tag{3.18}$$

Since by condition (σ_1) the values of the functions A, A'_*, D' have to be operators $E^1 \to E^N$, $E^N \to E^1$, $E^1 \to E^1$, respectively, i.e. of the "matrix" form

$$A = [u_1, \dots, u_N], \quad A'_* = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad D' = [w] \quad \text{with} \quad u_i, v_i, w \in H^\infty, \tag{3.19}$$

(c) immediately follows from (3. 19).

In [6] we did not ask whether for an inner Θ condition (c) is also sufficient for $S(\Theta)$ to be similar to $S(d)$ Now we shall show that it *is* sufficient, even for not inner Θ, if we add the condition

$$\dim \Delta(z) E^N \leq 1$$

which in our case is equivalent to ($*$).

Thus suppose (c) holds and write it in the form of a congruence modulo d in the algebra H^∞:

$$\sum_{i,j} u_i \omega_{ij} v_j \equiv 1 \pmod{d}. \tag{3.20}$$

By virtue of a well-known theorem in matrix theory we have

$$\omega_{ij}\omega_{kh} - \omega_{ih}\omega_{kj} = \pm d \cdot \det [\vartheta_{mn}]_{m \neq i,k;\, n \neq j,h}. \,^{*)} \tag{3.21}$$

From (3. 20) and (3. 21),

$$\omega_{kh} = \sum_{i,j} u_i \omega_{ij} \omega_{kh} v_j \equiv \sum_{i,j} u_i \omega_{ih} \omega_{kj} v_j \pmod{d}. \tag{3.22}$$

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*) See e. g. F. R. GANTMACHER, *Matrizenrechnung*. I (Berlin, 1958), p. 20, formula (33).

Thus there exist functions $t_{kh} \in H^\infty$ such that

(3. 23) $$\omega_{kh} = \sum_{i,j} \omega_{kj} v_j u_i \omega_{ih} + dt_{kh} \qquad (k, h = 1, \dots, N).$$

Let us set

$$A = [u_1, \dots, u_N], \quad A'_* = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad D = \begin{bmatrix} t_{11} \dots t_{1N} \\ \vdots \qquad \vdots \\ t_{N1} \dots t_{NN} \end{bmatrix}, \quad D' = [w]$$

and

$$A_* = A\Omega, \quad A' = \Omega A'_*.$$

On account of (3. 17) we have then

$$A_* \Theta = A\Omega\Theta = Ad = dA, \quad \Theta A' = \Theta\Omega A'_* = dA'_* = A'_* d,$$

i.e. conditions (α), (α') of Theorem 3. 1. On the other hand, condition (c) implies

(3. 24) $$\left.\begin{matrix} A_* A'_* + dD' \\ AA' + D'd \end{matrix}\right\} = A\Omega A'_* + dw = 1,$$

i.e. conditions (γ'_*) and (γ') of Theorem 3. 1.

Next observe that equation (3. 23) takes the forms

(3. 25) $$\Omega = \Omega A'_* A\Omega + dD = \begin{cases} \Omega A'_* A_* + dD, \\ A' A\Omega + dD; \end{cases}$$

multipling by Θ on the left or on the right and dividing by d we get

$$I_N = A'_* A_* + \Theta D \quad \text{and} \quad I_N = A'A + D\Theta,$$

i.e. conditions (γ_*) and (γ) of Theorem 3. 1. From (3. 25) we also derive, multiplying by A on the left or by A'_* on the right:

$$A\Omega = AA'A\Omega + AdD \quad \text{and} \quad \Omega A'_* = \Omega A'_* A_* A'_* + dDA'_*,$$

i.e.

$$(I - AA')A_* = dAD \quad \text{and} \quad A'(I - A_* A'_*) = dDA'_*.$$

Recalling (3. 24) and dividing by d we deduce from these equations that

$$D'A_* = AD \quad \text{and} \quad A'D' = DA'_*,$$

i.e. conditions (η) and (η') of Theorem 3. 1.

Therefore Theorem 3. 1 has the following

Corollary 2. *Let $\Theta(z)$ be an $N \times N$ matrix valued, purely contractive analytic function and suppose that* $\det \Theta(z) \not\equiv 0$. *Then condition* (c) *together with condition* $\dim \Delta(z)E^N \leqq 1$ *a.e., are necessary and sufficient for $S(\Theta)$ to be similar to* $S(\det \Theta)$.

4. Theorems on commutants

1. As an instructive application of Lemma 2. 1′ we are going to prove:

Theorem 4. 1. *Let T be a c.n.u. contraction on a separable Hilbert space, whose commutant $(T)'$ consists of operators $\varphi(T)$, where φ is a meromorphic function in the unit circle, of class N_T (cf.* [5], Chapter IV). *Then the characteristic function of T has only values isometries and coisometries, a.e. on the unit circle.*

Proof. It suffices to consider an operator $T=S(\Theta)$ generated by a purely contractive analytic function $\Theta(z)$, with values operators $\mathfrak{E}\to\mathfrak{E}_*$. Suppose that the set of points on the unit circle, where both $\Delta(z)$ and $\Delta_*(z)$ are non-zero, is of positive measure. As $\mathfrak{E}$ and $\mathfrak{E}_*$ are separable, this implies that there exist vectors $e\in\mathfrak{E}$ and $e_*\in\mathfrak{E}_*$ such that both $\Delta(z)e$ and $\Delta_*(z)e_*$ are non-zero on some set ω of positive measure. Consider the function E_0, with values operators $\mathfrak{E}_*\to\mathfrak{E}$, defined by

$$E_0(z)a=(a, \Delta_*(z)e_*)\Delta(z)e \quad \text{for} \quad a\in\mathfrak{E}_*.$$

It is bounded, measurable, and so is its restriction

$$E(z)=E_0(z)|\overline{\Delta_*(z)\mathfrak{E}_*},$$

which has values

$$E(z)\colon\ \overline{\Delta_*(z)\mathfrak{E}_*}\to\overline{\Delta(z)\mathfrak{E}}. \tag{4. 1}$$

Note that

$$(\Delta(z)E(z)\Delta_*(z)e_*, e)=|\Delta_*(z)e_*|^2|\Delta(z)e|^2\neq 0 \text{ on } \omega. \tag{4. 2}$$

Split ω into two disjoint sets of positive measure, say ω_1 and ω_2, and set

$$E_k(z)=i_k(z)E(z) \qquad (k=1,2), \tag{4. 3}$$

where $i_k(z)$ designates the indicator function of the set ω_k on the unit circle. The functions $E_k(z)$ also satisfy (4. 1) so we can apply Lemma 2. 1′ with $\Theta_1=\Theta_2=\Theta$, $\Delta_1=\Delta_2=\Delta$, $\Delta_{*1}=\Delta_{*2}=\Delta_*$, $A_*=0$, $A=0$, and E_k $(k=1, 2)$. Thus we obtain that the operators Y_k of multiplication by the functions

$$Y_k(z)=\begin{bmatrix}0 & 0\\ E_k(z)\Delta_*(z) & -E_k(z)\Theta(z)|\overline{\Delta(z)\mathfrak{E}}\end{bmatrix} \quad (k=1, 2)$$

belong to $\mathcal{I}^+(T, T)$. Let X_k denote the corresponding operators in $\mathcal{I}(T, T)$, i.e. in $(T)'$. We claim that $X_k\neq 0$. For if, e.g., $X_1=0$ then there exists, by Lemma 2. 2, a bounded analytic function $D_1(z)$ such that

$$\Theta(z)D_1(z)=0 \quad \text{and} \quad \Delta(z)D_1(z)=E_1(z)\Delta_*(z)$$

Hence, $D_1(z)=\Delta(z)E_1(z)\Delta_*(z)$ and, by (4. 2),

$$\big(D_1(z)e_*, e\big)=i_1(z)\big(\Delta(z)E(z)\Delta_*(z)e_*, e\big)\begin{cases}\neq 0 & \text{on} \quad \omega_1, \\ =0 & \text{on} \quad \omega_2.\end{cases}$$

But this is impossible on account of the analyticity of the function $\big(D_1(z)e_*, e\big)$.

By the hypothesis of the theorem there exist functions u_k, $v_k\in H^\infty$ $(k=1, 2)$ such that $v_k(T)$ is injective and

(4. 4) $$v_k(T)X_k-u_k(T) = 0.$$

As $X_k\neq 0$ we have $u_k(T)\neq 0$. Thus both $u_k(T)$ and $v_k(T)$ are non-zero, and therefore $u_k\neq 0$, $v_k\neq 0$.

Again using Lemma 2. 2 we infer from (4. 4) that there exist analytic functions $D_k(z)$ not necessarily the same as in the above argument such that

$$v_k(z)\begin{bmatrix}0 & 0 \\ E_k(z)\Delta_*(z) & -E_k(z)\Theta(z)|_z\end{bmatrix}-u_k(z)\begin{bmatrix}I_{\mathfrak{E}} & 0 \\ 0 & I_{\mathfrak{E}_*}|_z\end{bmatrix}=\begin{bmatrix}\Theta(z)D_k(z) & 0 \\ \Delta(z)D_k(z) & 0\end{bmatrix}$$

where $|_z$ denotes restriction to $\Delta(z)\mathfrak{E}$. Hence,

(i) $-u_k(z)I_{\mathfrak{E}_*}=\Theta(z)D_k(z)$,

(ii) $v_k(z)E_k(z)\Delta_*(z)=\Delta(z)D_k(z)$,

(iii) $\big(v_k(z)E_k(z)\Theta(z)+u_k(z)I_{\mathfrak{E}}\big)|_z = 0$.

From (i) we infer that $\Theta(z)D_k(z)$ commutes with every operator on $\mathfrak{E}_*$, in particular with $\Delta_*(z)$. As $\Theta(z)\Delta(z)=\Delta_*(z)\Theta(z)$ we deduce using (ii) that

$$\Theta D_k\Delta_*=\Delta_*\Theta D_k=\Theta\Delta D_k=v_k\Theta E_k\Delta_*,$$

and therefore

(4. 5) $$\Theta F_k=0 \quad \text{for} \quad F_k = D_k\Delta_*-v_kE_k\Delta_*.$$

On the other hand, (iii) and (4. 5) imply

$$u_k\Delta F_k =-v_kE_k\Theta\Delta F_k =-v_kE_k\Delta_*\Theta F_k = 0.$$

Since $u(z)\neq 0$ a.e., it follows

(4. 6) $$\Delta F_k=0.$$

Now (4. 5) and (4. 6) imply $F_k=0$, i.e. we have

(4. 7) $$D_k\Delta_*=v_kE_k\Delta_*;$$

again using (ii) we get

$$D_k \Delta_* = \Delta D_k \qquad (k=1, 2). \tag{4.8}$$

Setting

$$G = u_1 D_2 - u_2 D_1 \tag{4.9}$$

we have from (i): $\Theta G = 0$ while from (4. 8): $G\Delta_* = \Delta G$. Hence, $G\Delta_*^2 = \Delta^2 G$, $G\Theta\Theta^* = \Theta^*\Theta G = 0$, $G\Theta\Theta^* G^* = 0$, $\Theta^* G^* = 0$, $G\Theta = (\Theta^* G^*)^* = 0$, $G\Theta D_k = 0$, and by (i), $u_k G = 0$. Since $u_k(z) \neq 0$ a.e., we conclude: $G = 0$. Then, using (4. 7) and (4. 9),

$$(u_1 v_2 E_2 - u_2 v_1 E_1)\Delta_* = (u_1 D_2 - u_2 D_1)\Delta_* = G\Delta_* = 0,$$

and hence

$$(u_1 v_2 i_2 - u_2 v_1 i_1)E = 0.$$

As $E(z)$ is non-zero on ω, its factor must be zero there. But this factor equals $u_1(z)v_2(z)$ on ω_2, and we arrive at a contradiction to the fact that $u_1 v_2 \in H^\infty$ and $u_1 v_2 \neq 0$.

This contradiction proves the theorem.

Corollary. *Let T be as in Theorem* 4. 1 *and suppose, moreover, that its characteristic function $\Theta(z)$ has a scalar multiple $\delta \in H^\infty$, $\delta \neq 0$. Then T belongs to the class C_0; indeed, $\delta(T) = 0$.*

Proof. Since the function Θ has a scalar multiple, its values $\Theta(z)$ are boundedly invertible a.e. As an isometry or a coisometry is not invertible unless it is unitary we infer that $\Theta(z)$ is unitary a.e., and as a consequence $T \in C_{00}$ (i.e., $T^n \to 0$, $T^{*n} \to 0$). By [5], Theorem VI. 5. 1, we have then $\delta(T) = 0$.

2. Consider the c.n.u. contraction $T = S(\Theta)$ associated with a *scalar* valued purely contractive analytic function $\Theta(z)$ (i.e., $|\Theta(z)| \leqq 1$ and $\Theta(z)$ is not a constant of modulus 1). We shall show that if $\Theta(z) \not\equiv 0$ then $(T)'$ (i.e. $\mathscr{I}(T, T)$) is commutative. We shall even show that any two operators $Y_1, Y_2 \in \mathscr{I}^+(T, T)$ commute.

Let

$$\begin{bmatrix} A_{*k}(z) & 0 \\ B_k(z) & C_k(z) \end{bmatrix} \quad (k=1, 2)$$

be the corresponding matrix functions. As the entries are scalar valued functions, commutativity of Y_1 and Y_2 will be proved if we show that the function

$$F_{12}(z) = B_1(z) A_{*2}(z) + C_1(z) B_2(z)$$

is symmetric in the subscripts 1, 2. Since the values of $B_k(z)$ outside the set $\sigma = \{z : \Delta(z) \neq 0\}$ vanish (*cf.* condition (2. 2) in Lemma 2. 1) it suffices to consider

$F_{12}(z)$ on the set σ. Now by virtue of condition (2. 3) in Lemma 2. 1, we have

$$A_{*k}\Theta=\Theta A_k \quad \text{and} \quad B_k\Theta+C_k\Delta = \Delta A_k \qquad (k=1, 2).$$

Since $\Theta(z)$ cannot vanish on a set of positive measure we deduce from the first equation that $A_{*k}=A_k$, and from the second, that

$$F_{12}(z) = B_1(z)[B_2(z)\Theta(z)+C_2(z)\Delta(z)]/\Delta(z)+C_1(z)B_2(z) \quad \text{on } \sigma,$$

i.e.

$$F_{12}(z) = B_1(z)B_2(z)\Theta(z)/\Delta(z)+B_1(z)C_2(z)+C_1(z)B_2(z) \quad \text{on } \sigma,$$

and the symmetry in the subscripts 1, 2 is apparent.

The case $\Theta(z)\equiv 0$ is different. Consider in this case e.g. the matrices

$$Y_1(z)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad Y_2(z)=\begin{bmatrix}0 & 0\\ B(z) & 0\end{bmatrix},$$

where $B(z)$ is any scalar valued, *non-analytic* bounded measurable function. Both matrices satisfy conditions of Lemma 2. 1 (for $\Theta_1=\Theta_2=\Theta=0$ and $\Delta\equiv 0$), thus the corresponding operators Y_1, Y_2 belong to $\mathscr{I}^+(T, T)$. Then $X_1=\pi(Y_1)$ and $X_2=\pi(Y_2)$ belong to $(T)'$. By virtue of the Multiplication Property of the map π given in Sec. 1, we have $X_1X_2-X_2X_1 = \pi(Y_1Y_2-Y_2Y_1)$. Now the operator $Q = Y_1Y_2-Y_2Y_1$ is multiplication by the matrix function

$$Q(z) = \begin{bmatrix}0 & 0\\ -B(z) & 0\end{bmatrix},$$

and this is certainly not of the form

$$\begin{bmatrix}\Theta(z)D(z) & 0\\ \Delta(z)D(z) & 0\end{bmatrix}, \quad \text{i. e.} \quad \begin{bmatrix}0 & 0\\ D(z) & 0\end{bmatrix}$$

with analytic $D(z)$, and therefore, on account of Lemma 2. 2, $\pi(Q)\neq 0$. Thus X_1 and X_2 do not commute.

Observe that the characteristic function $\Theta(z)\equiv 0$ corresponds to an operator of the form

$$T = S\oplus S^*,$$

where S is a simple unilateral shift. That for such a T the commutant is not commutative can also be deduced from the fact proved in [7], Proposition 5, that there exists a non-zero operator X (indeed, a quasi-affinity) such that $S^*X=XS$.

So we have proved:

Theorem 4. 2. *Every c.n.u. contraction T with defect indices 1, 1 has a commutative commutant $(T)'$, with the only exception of the operator $T = S\oplus S^*$, where S is a simple unilateral shift.*

5. Inverse of a function of T

1. Let T be a c.n.u. contraction on the space $\mathfrak{H}$ and let V be its minimal isometric dilation on $\mathfrak{K}$ (we use the notations of Sec. 1). By the functional calculus developed in [5], Chapter III, the operators $u(T)$ and $u(V)$ have sense for every function $u \in H^\infty$ and are connected by the relation

$$u(T) = P_{\mathfrak{H}} u(V)|\mathfrak{H}. \tag{5. 1}$$

If R is the unitary part of V then $u(R)$ also has sense (it is the restriction of $u(V)$ to $\mathfrak{R}$).

Theorem 5. 1. *If $u(T)$ is boundedly invertible then so is $u(R)$ and we have*

$$\|u(R)^{-1}\| \leqq \|u(T)^{-1}\|. \tag{5. 2}$$

Proof. We use the fact, proved in the proof of Proposition II. 6. 2 in [5], that for every $k \in \mathfrak{R}$ there exists a sequence of elements $h_n \in \mathfrak{H}$ such that

$$k = \lim_{n \to \infty} V^n h_n.$$

This implies:

$$\|u(R)k\| = \|u(V)k\| = \lim \|u(V)V^n h_n\| = \lim \|V^n u(V) h_n\| =$$

$$= \lim \|u(V)h_n\| \geqq \lim\inf \|u(T)h_n\| \geqq c \lim\inf \|h_n\| = c\|k\|,$$

where $c = \|u(T)^{-1}\|^{-1}$. As $u(R)$ is normal we conclude that $u(R)$ is boundedly invertible and (5. 2) holds.

2. Thus the existence of $u(R)^{-1}$ is necessary for the existence of $u(T)^{-1}$. Necessary and sufficient conditions follow from results of Sec. 1 when we observe that (1. 3) implies $u(T)P_{\mathfrak{H}} = P_{\mathfrak{H}} u(V)$ so that on account of (1. 5) we have $u(V) \in \mathscr{I}^+(T, T)$ and $\pi(u(V)) = u(T)$. Using matrices corresponding to the decomposition $\mathfrak{K} = \mathfrak{S}_* \oplus \mathfrak{R}$ we deduce from Lemmas 1. 1, 1. 2, and the Multiplication Property of the map π, that $u(T)$ is boundedly invertible if and only if there exist operators

$A_* \in \mathscr{I}(S_*, S_*)$, $A \in \mathscr{I}(S, S)$, $B \in \mathscr{I}(S_*, R)$, $C \in \mathscr{I}(R, R)$, $D \in \mathscr{I}(S_*, S)$, $D' \in \mathscr{I}(S_*, S)$

satisfying the equations

$$(\alpha) \quad A_* \hat{\Theta} = \hat{\Theta} A, \qquad (\beta) \quad B\hat{\Theta} + C\hat{\Delta} = \hat{\Delta} A,$$

$$(\varkappa)\begin{cases} \begin{bmatrix} A_* & 0 \\ B & C \end{bmatrix} \begin{bmatrix} u(S_*) & 0 \\ 0 & u(R) \end{bmatrix} = I - \begin{bmatrix} \hat{\Theta} D & 0 \\ \hat{\Delta} D & 0 \end{bmatrix}, \\ \begin{bmatrix} u(S_*) & 0 \\ 0 & u(R) \end{bmatrix} \begin{bmatrix} A_* & 0 \\ B & C \end{bmatrix} = I - \begin{bmatrix} \hat{\Theta} D' & 0 \\ \hat{\Delta} D' & 0 \end{bmatrix}. \end{cases}$$

As a consequence of the intertwining properties of A_*, B and C we can take $D'=D$ and condition $(\varkappa)$ is equivalent to the following system of conditions:

$$(\varkappa_1)\quad A_* u(S_*)+\hat{\Theta} D = I, \qquad (\varkappa_2)\quad B u(S_*)+\hat{\Delta} D=0, \qquad (\varkappa_3)\quad C=u(R)^{-1}.$$

Taking into account that (1. 14) implies

$$u(S_*)\hat{\Theta}=\hat{\Theta}\, u(S) \quad \text{and} \quad u(R)\hat{\Delta}=\hat{\Delta}\, u(S)$$

we deduce:

$$\hat{\Theta}(A\, u(S)+D\hat{\Theta}-1)\overset{\alpha}{=}(A_* u(S_*)+\hat{\Theta} D-I)\hat{\Theta}\overset{\varkappa_1}{=}0,$$

$$\hat{\Delta}(A\, u(S)+D\hat{\Theta}-I)\overset{\varkappa_{2,3}}{=}\hat{\Delta}A\, u(S)-B\, u(S_*)\hat{\Theta}-C\, u(R)\hat{\Delta}=(\hat{\Delta}A-B\hat{\Theta}-C\hat{\Delta})u(S)\overset{\beta}{=}0,$$

and hence

$$(\lambda)\qquad A\, u(S)+D\hat{\Theta} = I.$$

Conversely, (β) and $(\varkappa_2)$ are implied by (λ) and the rest of the conditions if we set

$$B = -C\hat{\Delta} D.$$

Indeed, the intertwining property for B follows immediately from those for C, $\hat{\Delta}$, and D, while $(\varkappa_2)$ follows from the equations

$$B\, u(S_*) = -C\hat{\Delta} D\, u(S) = -C\hat{\Delta}\, u(S) D = -C\, u(R)\hat{\Delta} D = -\hat{\Delta} D;$$

finally, (β) follows from the equations

$$u(R)(B\hat{\Theta}+C\hat{\Delta}-\hat{\Delta}A) = -\hat{\Delta}D\hat{\Theta}+\hat{\Delta}-\hat{\Delta}\, u(S)A = \hat{\Delta}(-D\hat{\Theta}+I-u(S)A)\overset{\lambda}{=}0$$

when we multiply by C on the left.

Thus the initial set of conditions can be replaced by the set (α), $(\varkappa_1)$, $(\varkappa_3)$, (λ). Multiplying $(\varkappa_1)$ and (λ) by $\hat{\Theta}$ on the right and on the left, respectively, and using the intertwining properties and substracting we obtain that

$$u(S_*)(A_*\hat{\Theta}-\hat{\Theta}A)=0.$$

As the unilateral shift S_* is the restriction of a bilateral shift U, and hence $u(S_*)$ is a restriction of $u(U)$, and as $u(U)$ has zero null-space for $u\neq 0$ (because then $u(z)\neq 0$ a.e.), we conclude that (α) also holds, i.e. it is a consequence of $(\varkappa_1)$ and (λ).

So we have proved:

Theorem 5. 2. *Let $u\in H^\infty$, $u\neq 0$. In order that $u(T)$ be boundedly invertible it is necessary and sufficient that*

a) *$u(R)$ be boundedly invertible,*
b) *there exist operators $A_*\in\mathscr{I}(S_*, S_*)$, $A\in\mathscr{I}(S, S)$, $D\in\mathscr{I}(S_*, S)$ such that*

$$A\, u(S)+D\hat{\Theta} = I_{\mathfrak{S}}, \quad A_* u(S_*)+\hat{\Theta} D = I_{\mathfrak{S}*}.$$

Remark. Since $u(R)$ is normal, condition a) is equivalent to the condition that

$$\|u(R)f\| \geqq m\|f\| \quad \text{for some} \quad m>0 \quad \text{and all} \quad f\in\mathfrak{R}.$$

3. If $T=S(\Theta)$, Θ being a purely contractive analytic function with values operators $\Theta(z):\mathfrak{E}\to\mathfrak{E}_*$, then the above conditions a), b) can be expressed in the following form:

a) $|u(z)|\geqq m>0$ at a.e. point z where $\Delta(z)\neq 0$, i.e. $\Theta(z)$ is not an isometry,

b) there exist bounded analytic functions A_*, A, D with values operators

$$A_*(z)\colon\ \mathfrak{E}_*\to\mathfrak{E}_*,\quad A(z)\colon\ \mathfrak{E}\to\mathfrak{E},\quad D(z)\colon\ \mathfrak{E}_*\to\mathfrak{E}\quad \text{a.e.}$$

such that

$$u(z)A(z)+D(z)\Theta(z) = I_{\mathfrak{E}},\quad u(z)A_*(z)+\Theta(z)D(z) = I_{\mathfrak{E}_*}\quad \text{a.e.} \tag{5.3}$$

Now $u(T)$ is boundedly invertible if and only if so is $u(T)^*$; and $u(T)^*$ is unitarily equivalent to $u^{\tilde{}}(T')$, where $T'=S(\Theta^{\tilde{}})$. Here we use the notations $u^{\tilde{}}$ and $\Theta^{\tilde{}}$ for the functions defined by

$$u^{\tilde{}}(z)=\overline{u(\bar{z})},\quad \tilde{\Theta}(z)=\Theta(\bar{z})^*$$

(*cf.* [5], Theorem III. 2. 1 and Chapter VI).

Thus conditions a), b) imply that $|u^{\tilde{}}(z)|\geqq m_*>0$ at a.e. point z where $\Theta^{\tilde{}}(z)$ is not an isometry, i.e. $|u(z)|\geqq m_*$ at a.e. point z where $\Theta(z)$ is not a coisometry. Hence, a), b) imply that $|u(z)|\geqq p(>0)$ at a.e. point z where $\Theta(z)$ is not unitary.

So we have:

Theorem 5. 3. *Let $T=S(\Theta)$ and $u\in H^\infty$, $u\neq 0$. In order that $u(T)$ be boundedly invertible it is necessary and sufficient that there exist bounded analytic functions A_*, A, D satisfying conditions* (5. 3), *and a positive number p such that*

$$|u(z)|\geqq p \quad \textit{at a.e. point} \quad z=e^{it} \quad \textit{where} \quad \Theta(z) \quad \textit{is not unitary.} \tag{5.4}$$

4. Consider the particular case when $\Theta(z)$ is an $N\times N$ matrix valued function, limit on the unit circle of a (purely contractive, analytic) function $\Theta(\lambda)$ on the open unit disc. Let $d(\lambda)=\det\Theta(\lambda)$.

As a contraction on a finite dimensional euclidean space is unitary if and only if its determinant is of absolute value 1, condition (5. 4) can be expressed in the form

$$|u(z)|\geqq p \text{ at a.e. point } z \text{ where } |d(z)|\neq 1. \tag{5.4'}$$

Next we notice that the equations (5. 3) hold in the unit disc as well. Thus at every point λ where $\Theta(\lambda)$ has a bounded inverse we have

$$\Theta(\lambda)^{-1} = u(\lambda)A(\lambda)\Theta(\lambda)^{-1}+D(\lambda),$$

and hence

$$(5.5) \qquad |\Theta(\lambda)^{-1}| \leqq M(|u(\lambda)|\,|\Theta(\lambda)^{-1}|+1), \quad \text{or} \quad |u(\lambda)|+|\Theta(\lambda)^{-1}|^{-1} \geqq 1/M$$

where M equals the larger one of the values $\|A\|_\infty$ and $\|D\|_\infty$. As for every invertible operator Z on E^N we have $|\det Z|^N \geqq |Z^{-1}|^{-1}$ (*cf.* Lemma 2. 3 in [1]), inequality (5. 5) implies

$$|u(\lambda)|+|d(\lambda)|^{1/N} \geqq 1/M,$$

and hence

$$(5.6) \qquad |u(\lambda)|+|d(\lambda)| \geqq q\,(>0).$$

If $d(\lambda)\not\equiv 0$ then $\Theta(\lambda)^{-1}$ exists at every point λ of the open unit disc, perhaps with the exception of countably many points, therefore (5. 6) holds then everywhere in the unit disc. By virtue of the "Corona Theorem" condition (5. 6) is equivalent to the existence of functions $a, b \in H^\infty$ such that

$$(5.7) \qquad u(\lambda)a(\lambda)+d(\lambda)b(\lambda) = 1.$$

Conversely, (5. 7) implies equations (5. 3), with $A(\lambda)=A_*(\lambda)=a(\lambda)I_N$, and $D(\lambda)=$ $=b(\lambda)\Omega(\lambda)$, where $\Omega(\lambda)$ designates the algebraic adjoint of the matrix $\Theta(\lambda)$.

We state our result as follows:

Theorem 5. 4. *Let $\Theta(\lambda)$ be a purely contractive analytic $N\times N$ matrix function with $d(\lambda)=\det\Theta(\lambda)\not\equiv 0$, and let $T=S(\Theta)$ and $u\in H^\infty$. The operator $u(T)$ is boundedly invertible if and only if there exist constants $p, q>0$ such that*

α) $|u(z)|\geqq p$ *at a.e. point* $z=e^{it}$ *where* $|d(z)|\neq 1$, *and*

β) $|u(\lambda)|+|d(\lambda)| \geqq q$ *at every point* λ, $|\lambda|<1$.

The particular case of this theorem when $\Theta(\lambda)$ is an *inner* function, was considered in FUHRMANN [1]. Let us add that another generalization of Fuhrmann's result was given in HERRERO [2].

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