On models for noncontractions

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1. Introduction

1.1. Characteristic functions. The characteristic operator function Θ_T of a bounded linear operator T on a Hilbert space $\mathfrak{H}$ is by definition the operator-valued analytic function

(1) $$\Theta_T(z) = TJ_T - zQ_*(I - zT^*)^{-1}Q$$

where $J_T = \operatorname{sgn}(I - T^*T)$, $Q = |I - T^*T|^{\frac{1}{2}}$ and $Q_* = |I - TT^*|^{\frac{1}{2}}$, in the sense of the self-adjoint operator calculus (here $\operatorname{sgn} 0 = 1$), and where Θ_T acts from $\bar{R}(Q)$, the closure of the range of Q, to $\bar{R}(Q_*)$.

If T is a contraction, so that the operator J_T (and the absolute value signs) disappear from (1), Θ_T has been studied quite a bit and is fairly well understood. SZ.-NAGY and FOIAŞ, for example, in their book [6], study the relationship of T and Θ_T. Basic to their theory is the construction of a "canonical model" — a contraction operator T of a canonical type — such that $\Theta = \Theta_T$, for a given analytic operator function Θ with $\|\Theta(z)\| \leq 1$ for $|z| < 1$.

Several recent papers have concerned more general $\Theta(z)$; see, for example, KUŽEL' [4] and DAVIS and FOIAŞ [3]. BRODSKII, GOHBERG and KREIN [2], working with a characteristic operator function somewhat different from (1), have given necessary and sufficient conditions that an analytic operator-valued function Θ should have the form $\Theta = \Theta_T$, for some bounded (invertible) operator T. Their condition translates into Theorem 1 below. Their proof uses Neumark's Theorem and does not appear to provide a clear analogue of the Sz.-Nagy—Foiaş model theory.

In this paper we give a construction (Theorem 2 below) which, although less geometrical than that of Sz.-Nagy and Foiaş, does yield a model analogous to theirs and also contains the theorem of Brodskii, Gohberg and Krein (Theorem 1) as a corollary.

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1.2. Statement of results. More precisely, let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces of the same dimension, let $B(z)$ be a function whose values are bounded operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, and let $J=\operatorname{sgn}(I-B(0)^*B(0))$ and $J_*=\operatorname{sgn}(I-B(0)B(0)^*)$. The conditions of Brodskii, Gohberg and Krein, applied to our characteristic operator function become

Theorem 1. ([2], Theorem 6.1.) *Suppose $B(z)$ is analytic in some neighborhood D of 0. Then B is the characteristic operator function of some invertible operator if and only if B satisfies*

(i) *$B(0)$ is invertible,*

(ii) *the operator valued function*

$$G(z) = [\mathfrak{U}^*+B(z)]^{-1}[\mathfrak{U}^*-B(z)]J,$$

where $\mathfrak{U}\colon \mathfrak{H}_2\to\mathfrak{H}_1$ is a unitary operator satisfying $\mathfrak{U}J_*=J\mathfrak{U}$, extends to be analytic in $|z|<1$ with positive real part there:

$$\operatorname{Re}(G(z)\,x, x) \geqq 0 \quad \text{if} \quad |z|<1 \quad \text{and} \quad x\in\mathfrak{H}_1.$$

The existence of the unitary operator $\mathfrak{U}$ in (ii) comes from the polar representation of the (invertible) operator $B(0)$.

Theorem 1 will be seen to follow from

Theorem 2. *Let $B(z)$ be analytic and invertible in an open set D, with $0\in D\subset \subset\{|z|<1\}$. Extend $B(z)$ to the reflection $\tilde{D}$ of D by defining*

$$B(z) = J_*B(\bar{z}^{-1})^{*-1}J.$$

Then $B(z)$ is a characteristic operator function if and only if

$$b(w, z) = (1-\bar{w}z)^{-1}[J_*-B(z)JB(w)^*]$$

is a positive definite operator function on $\mathfrak{H}_2$.

The condition on $b(z, w)$ means that for $z_1, \dots, z_n\in D\cup\tilde{D}$ and for $x_1, \dots, x_n\in\mathfrak{H}_2$, not all 0, we have

$$\sum (b(z_i, z_j)x_i, x_j) > 0. \tag{2}$$

1.3. Remarks on the theorems. The proofs of (the sufficiency parts of) the theorems will be given in Section 2 (Theorem 1) and Sections 3—5 (Theorem 2). The necessity parts are less difficult and will be proved in the next section.

We shall continually use the following fact about the Q's and J's. Since $(I-T^*T)T^*=T^*(I-TT^*)$ it follows that $f(I-T^*T)T^*=T^*f(I-TT^*)$ for any (bounded, Borel) function f. From this there follow relations of the form $JB(0)^*= =B(0)^*J_*$, $Q_*B(0)=B(0)Q$, etc.

As we have pointed out, a different characteristic function is used in [2]. Let $K=(\Theta_T(0)^*\Theta_T(0))^{\frac{1}{2}}$, so that $\Theta_T(0)=\mathfrak{U}^*K$. Then

$$\Theta_T(z) = \Theta(z) = \Theta(0)^{*-1}J[J\Theta(0)^*\Theta(z)] = \mathfrak{U}^*K^{-1}J[\Theta(0)^*J_*\Theta(z)],$$

and from a relation of KUŽEL' [4], this is

$$= \mathfrak{U}^*K^{-1}J[J-Q(I-zT^*)^{-1}Q] = \mathfrak{U}^*\Theta_N(z)$$

where Θ_N is the characteristic function of the „node" $(\mathfrak{H}, \mathfrak{H}_1; T, Q, J)$; [2].

As with contractions, if $\Theta_1=U\Theta V$, where U and V are constant isometries, then Θ_1 and Θ_2 are considered the same, as characteristic functions. Thus, given $B(z)$, one need only prove the existence of a T such that $B=U\Theta_T V$. In an appendix (Section 6) we have included our own proof that if S and T are (invertible) bounded operators and $\Theta_S=U\Theta_T V$, then S and T are unitarily equivalent.

1.4. Proofs of necessity. The proof of necessity in Theorem 2 follows easily from a relation of KUŽEL' [4]:

$$J_*-\Theta_T(z)J\Theta_T(w)^* = (1-z\bar{w})Q_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_* \tag{3}$$

so that

$$b(w, z) = Q_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_*$$

and this implies that $b(w, z)$ is a positive definite operator function.

To prove necessity in Theorem 1, we refer to the corresponding proof in [2]. Actually (i) is evident from (1); only (ii) needs attention. We have that $\Theta_T(z)=\mathfrak{U}^*\Theta_N(z)$, as in Section 1.3 above. Now, in the notation of [2, Section 6], it is easily seen that $\Theta_N(0)=K$ and so $H_0=K$, $U_0=I$. Thus

$$G(z) = [\mathfrak{U}^*+\Theta_T(z)]^{-1}[\mathfrak{U}^*-\Theta_T(z)]J = [I+\mathfrak{U}\Theta_T(z)]^{-1}[I-\mathfrak{U}\Theta_T(z)]J =$$
$$= [I+U_0^{-1}\Theta_N(z)]^{-1}[I-U_0^{-1}\Theta_N(z)]J = J\Omega(z)J$$

and the necessity part of Theorem 1 follows from that of [2, Theorem 6.1].

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2. Proof of Theorem 1, assuming Theorem 2

2.1. Integral representation. The function $G(z)$ is analytic for $|z|<1$ and $\operatorname{Re} G(z)\geqq 0$ in $|z|<1$. Thus it follows from the operator-valued Riesz—Herglotz Theorem [1, p. 84] that there is a positive, operator valued measure dF such that

$$G(z) = \int_0^{2\pi} [e^{i\theta}+z]/[e^{i\theta}-z]\,dF(\theta). \tag{4}$$

Using this, we obtain

$$G(z)+G(w)^* = \int_0^{2\pi} \{[e^{i\theta}+z]/[e^{i\theta}-z]+[e^{-i\theta}+\bar{w}]/[e^{-i\theta}-\bar{w}]\}\, dF(\theta) =$$

$$= \int_0^{2\pi} [1-z\bar{w}][(1-e^{-i\theta}z)(1-\bar{w}e^{i\theta})]^{-1} dF(\theta).$$

Computing $G(z)+G(w)^*$ another way, using the definition of $G(z)$, we get

$$[\mathfrak{U}^*+B(z)][G(z)+G(w)^*][\mathfrak{U}+B(w)^*] =$$

$$= [\mathfrak{U}^*-B(z)]J[\mathfrak{U}+B(w)^*]+[\mathfrak{U}^*+B(z)]J[\mathfrak{U}-B(w)^*] =$$

$$= 2[\mathfrak{U}^*J\mathfrak{U}-B(z)JB(w)^*] = 2[J_*-B(z)JB(w)^*].$$

Combining this with the first expression for $G(z)+G(w)^*$ gives

$$b(w,z) = \frac{1}{2}[\mathfrak{U}^*+B(z)]\Big(\int_0^{2\pi} [(1-e^{-i\theta}z)(1-\bar{w}e^{i\theta})]^{-1} dF(\theta)\Big)[\mathfrak{U}+B(w)^*].$$

2.2. Operator integrals. We have thus far integrated only scalars against operator measures; we need now some notation for the integration of operator valued functions against them. Let $E(t)$ and $H(t)$ be operator-valued functions and $dF(t)$ a positive operator-valued measure on $[0, 2\pi]$. Suppose that $E(t)$ and $H(t)$ are the boundary values of operator-valued functions, holomorphic in $|z|\leqq 1$ or, more generally, that $H(t)$ is holomorphic and $E(t)$ is equal to a continuous (scalar-valued) function times an analytic function. Then, according to LANGER [5, Lemma 1″], the integral

$$\int_0^{2\pi} E(t)\big(dF(t)\big)H(t), \tag{5}$$

defined in terms of the convergence of Riemann sums of the form

$$\sum E(\xi_i)[F(x_i)-F(x_{i-1})]H(\xi_i),$$

exists. We shall use the integral (5) in case $E(t)$ is a linear combination of continuous functions times constant operator functions. Clearly one has:

i) For T a constant operator,

$$T\int_0^{2\pi} E(t)\,dF(t)H(t) = \int_0^{2\pi} TE(t)\,dF(t)H(t),$$

$$\Big[\int_0^{2\pi} E(t)\,dF(t)H(t)\Big]T = \int_0^{2\pi} E(t)\,dF(t)[H(t)T].$$

ii)
$$\int_0^{2\pi} E(t)\,dF(t)E(t)^* \geqq 0.$$

It follows that we may rewrite the last integral in Section 2.1 as

$$b(w, z) = \frac{1}{2} \int_0^{2\pi} [\mathfrak{U}^* + B(z)][1 - e^{-i\theta} z]^{-1} dF(\theta) [\mathfrak{U} + B(w)^*][1 - e^{i\theta} \bar{w}]^{-1},$$

for $|z|, |w| < 1$.

2.3. Positive definiteness of $b(w, z)$ in $|z|, |w| < 1$. Let $z_1, \ldots, z_n$ be complex numbers in D, $j = 1, \ldots, n$ and let $x_1, \ldots, x_n \in \mathfrak{H}_2$. Let $x_i^* x$ denote the linear functional $x \to (x, x_i)$ on $\mathfrak{H}_2$. We have

$$\sum_{i,j} (b(z_i, z_j) x_i, x_j) =$$

$$= \frac{1}{2} \sum_{i,j} x_j^* \int_0^{2\pi} (\mathfrak{U}^* + B(z_j)) (1 - e^{-i\theta} z_j)^{-1} dF(\theta) (\mathfrak{U} + B(z_i)^*) (1 - e^{i\theta} \bar{z}_i)^{-1} x_i =$$

$$= \frac{1}{2} \sum_{i,j} \int_0^{2\pi} [(\mathfrak{U} + B(z_j)^*) (1 - e^{i\theta} \bar{z}_j)^{-1} x_j]^* dF(\theta) (\mathfrak{U} + B(z_i)^*) (1 - e^{i\theta} \bar{z}_i)^{-1} x_i =$$

$$= \frac{1}{2} \int_0^{2\pi} K(\theta)^* dF(\theta) K(\theta) \geqq 0,$$

where $K(\theta) = \sum_i (\mathfrak{U} + B(z_i)^*)(1 - e^{i\theta} \bar{z}_i)^{-1} x_i$. This proves $b(w, z)$ is a positive definite operator function in D.

Actually there is one difficulty here: the fact that the above sum has only been proved to be nonnegative. The reader will see, however, that he may easily divide out any elements of 0 norm in the proof of Theorem 2.

2.4. Positive definiteness of $b(w, z)$ in general. To complete the proof of the positive definiteness of $b(w, z)$, we shall prove that the integral representation (4) persists in $|z| > 1$. Once this is done, the proof in Sections 2.2 and 2.3 will apply *verbatim* to the case where one or both of the variables z, w has modulus >1.

To extend (4) to $|z| > 1$, we shall prove that $G(z)$ satisfies

$$G(\bar{z}^{-1})^* = -G(z). \tag{6}$$

Since the right side of (4) obviously satisfies the analogous functional equation, the proof will be complete when (6) is verified.

To prove (6), we substitute in the definition of G and use the extension of $B(z)$ to $|z| > 1$.

$$G(\bar{z}^{-1})^* = \{[\mathfrak{U}^* + B(\bar{z}^{-1})]^{-1} [\mathfrak{U}^* J - B(\bar{z}^{-1}) J]\}^* = [J\mathfrak{U} - JB(\bar{z}^{-1})^*][\mathfrak{U} + B(\bar{z}^{-1})^*]^{-1} =$$

$$= [J\mathfrak{U} - B(z)^{-1} J_*][\mathfrak{U} + JB(z)^{-1} J_*]^{-1} = [\mathfrak{U}B(z) - I] B(z)^{-1} J_* J_* B(z) [J\mathfrak{U}B(z) + J]^{-1} =$$

$$= [\mathfrak{U}B(z) - I][\mathfrak{U}B(z) + I]^{-1} J = [\mathfrak{U}B(z) + I]^{-1} [\mathfrak{U}B(z) - I] J =$$

$$= [B(z) + \mathfrak{U}^*]^{-1} [B(z) - \mathfrak{U}^*] J = -G(z)$$

and this completes the proof.

3. The Hilbert space H and the operator S

3.1. Definition of H. We now suppose only that $B(z)$ is analytic in a neighborhood D of 0, and that $b(w, z)$ is a positive definite operator function. Let $\tilde{D}$ denote the reflection of D, i.e. $\tilde{D}=\{\bar{z}^{-1}: z\in D\}$, and let H^0 be a set indexed by $(D\cup\tilde{D})\times\times\mathfrak{H}_2$; elements of H^0 are written $k_z f$ where $z\in D\cup\tilde{D}$ and $f\in\mathfrak{H}_2$. H^1 is defined to be the set of all finite linear combinations of elements of H^0.

We give H^1 the structure of a pre-Hilbert space by defining

$$(k_z f, k_w g) = \big(b(z, w)f, g\big) = (1-w\bar{z})^{-1}\big([J_*-B(w)JB(z)^*]f, g\big).$$

The positive-definiteness of $b(z, w)$ implies that this is a *bona fide* inner product on H^1. The Hilbert space H is the completion of H^1 in this norm.

3.2. The subspace $h_0 H$. Let R_0 be a real number, so large that $\{x: x\geqq R_0\}$ lies in $\tilde{D}$. Let $f\in\mathfrak{H}_1$. We claim that

$$h_0 f = \lim_{R_0\leqq R\to\infty} Rk_R J_* B(0)f \tag{7}$$

exists in H.

To prove the claim, pick $M, R\geqq R_0$ and compute

$$\begin{aligned}
&\|Rk_R J_* B(0)f - Mk_M J_* B(0)f\|^2 =\\
&= R^2\big(b(R, R)J_*B(0)f, J_*B(0)f\big) - 2RM\,\mathrm{Re}\big(b(R, M)J_*B(0)f, J_*B(0)f\big)+\\
&\quad + M^2\big(b(M, M)J_*B(0)f, J_*B(0)f\big) =\\
&= [R^2(1-R^2)^{-1} - 2RM(1-RM)^{-1} + M^2(1-M^2)^{-1}]\big(J_*B(0)f, B(0)f\big) -\\
&\quad - R^2(1-R^2)^{-1}\big(JB(R)^*J_*B(0)f, B(R)^*J_*B(0)f\big)+\\
&\quad + 2RM(1-RM)^{-1}\big(JB(R)^*J_*B(0)f, B(M)^*J_*B(0)f\big)-\\
&\quad - M^2(1-M^2)^{-1}\big(JB(M)^*J_*B(0)f, B(M)^*J_*B(0)f\big).
\end{aligned}$$

Elementary calculus shows that the first term tends to 0 as $M, R\to\infty$. Since $B(R)$ tends uniformly to $B(\infty)=J_* B(0)^{*-1}J$ as $R\to\infty$, it is not hard to see that the sum of the last three terms tends to 0 and we have proved the existence of the limit (7).

Now we want to find $(h_0 f, k_z g)$ for $z\in D\cup\tilde{D}$. All this takes is an application of (7). In fact

$$\begin{aligned}
(h_0 f, k_z g) &= \lim_{R\to\infty} R\big(k_R J_* B(0)f, k_z g\big) =\\
&= \lim_{R\to\infty} R(1-Rz)^{-1}\big([J_* - B(z)JB(R)^*]J_*B(0)f, g\big) =\\
&= \lim_{R\to\infty} (R^{-1}-x)^{-1}\big([B(0) - B(z)JB(R)^*J_*B(0)]f, g\big) =\\
&= (-z)^{-1}\big([B(0) - B(z)JB(\infty)^*J_*B(0)]f, g\big)
\end{aligned}$$

and we have

$$(8)\qquad (h_0 f, k_z g) = z^{-1}([B(z)-B(0)]f, g).$$

3.3. The operator S. We define an operator S on the dense subset H^1 of H by

$$Sk_z f = \bar{z} k_z f - h_0 JB(z)^* f,$$

for $z \in D \cup \tilde{D}$ and $f \in \mathfrak{H}_2$. In part 4 of this paper, we shall show that S extends (uniquely) to a bounded operator (also denoted S) on H and that the operator S^* has $B(z)$ as its characteristic operator function.

4. Boundedness of S

4.1. Proposition. *For $z, w \in D \cup \tilde{D}$, S satisfies*

$$(Sk_z f, k_w g) = (k_z f, \bar{w}^{-1}[k_w - k_0]g).$$

Thus, the domain of the adjoint S^* of S contains H^1 (and hence is dense in H) and satisfies

$$(9)\qquad S^* k_z f = \bar{z}^{-1}[k_z - k_0]f.$$

In particular, S has a closure.

Proof.

$$(Sk_z f, k_w g) = (\bar{z} k_z f - h_0 JB(z)^* f, k_w g) =$$
$$= \bar{z}(1-\bar{z}w)^{-1}([J_* - B(w)JB(z)^*]f, g) - w^{-1}([B(w)-B(0)]JB(z)^* f, g) =$$
$$= w^{-1}[(1-\bar{z}w)^{-1} - 1]([J_* - B(w)JB(z)^*]f, g) - w^{-1}([B(w)-B(0)]JB(z)^* f, g) =$$
$$= w^{-1}(k_z f, k_w g) - w^{-1}([J_* - B(w)JB(z)^*]f, g) -$$
$$- w^{-1}([B(w)JB(z)^* - B(0)JB(z)^*]f, g) =$$
$$= w^{-1}(k_z f, k_w g) - w^{-1}([J_* - B(0)JB(z)^*]f, g) = w^{-1}(k_z f, k_w g) - w^{-1}(k_z f, k_0 g).$$

This proves the first part of the proposition. All the other parts follow at once.

Henceforth, S will denote the closure of the operator S in Section 3.3 above.

4.2. Proposition. *For $z \in D \cup \tilde{D}$ and $f \in \mathfrak{H}_2$,*

$$(I - SS^*)k_z f = \bar{z}^{-1} h_0 J[B(z)^* - B(0)^*]f.$$

Thus $I - SS^* = 0$ on $(h_0 \mathfrak{H}_1)^\perp$, so that $(h_0 \mathfrak{H}_1)^\perp$ is contained in the domain of S^*.

Proof.

$$(I - SS^*)k_z f = k_z f - S\bar{z}^{-1}[k_z - k_0]f =$$
$$= k_z f - \bar{z}^{-1}[\bar{z} k_z f - h_0 JB(z)^* f + h_0 JB(0)^* f] = \bar{z} h_0 J[B(z)^* - B(0)^*]f.$$

Again, all the other claims are obvious.

4.3. Proposition. *$h_0\mathfrak{H}_1$ is contained in the domain of S^* and*

$$S^* h_0 f = -k_0 J_* B(0) f.$$

Proof. Fix $f\in\mathfrak{H}_1$. We know $h_0 f$ is the limit of $Rk_R J_* B(0)f$. Let us compute

$$\lim_{R\to\infty} S^* Rk_R J_* B(0)f = \lim_{R\to\infty} RR^{-1}[k_R - k_0] J_* B(0) f =$$
$$= \lim_{R\to\infty} [k_R - k_0] J_* B(0) f = -k_0 J_* B(0) f,$$

where $k_R J_* B(0)f$ converges to 0 since the limit in (7) exists. Now the proposition follows from the fact that S^* is closed.

4.4. Lemma. *The following three operators are isometries:*

$$L_1: \mathfrak{H}_2 \to k_0\mathfrak{H}_2, \quad L_2: \mathfrak{H}_1 \to h_0\mathfrak{H}_1, \quad L_3: k_0\mathfrak{H}_2 \to k_0\mathfrak{H}_2$$

where

$$L_1(|J_* - B(0)JB(0)^*|^{1/2} f) = k_0 f,$$
$$L_2(f) = -h_0|J - B(0)^* J_* B(0)|^{-1/2} f.$$
$$L_3(k_0 f) = k_0 J_* f.$$

Proof. For $f\in\mathfrak{H}_2$,

$$\||J_* - B(0)JB(0)^*|^{1/2} f\|^2 = ([J_* - B(0)JB(0)^*]f, f) = \|k_0 f\|^2,$$

so L_1 is an isometry.

For L_2, we have to determine the norm of $h_0 g$, for $g\in\mathfrak{H}_1$. We have

$$\|h_0 g\|^2 = \lim_{R\to\infty} \|Rk_R J_* B(0) g\|^2 =$$
$$= \lim_{R\to\infty} R^2(1 - R^2)^{-1}([J_* - B(R)JB(R)^*] J_* B(0) g, J_* B(0) g) =$$
$$= ([J - B(0)^* J_* B(0)] g, g).$$

Now we can compute

$$\|L_2 f\|^2 = ([J - B(0)^* J_* B(0)] [J - B(0)^* J_* B(0)]^{-1/2} f, [J - B(0)^* J_* B(0)]^{-1/2} f) = \|f\|^2$$

so L_2 is an isometry.

Finally, let $f\in\mathfrak{H}_2$. We have

$$\|k_0 J_* f\|^2 = ([J_* - B(0)JB(0)^*] J_* f, J_* f) =$$
$$= ([J_* - J_* B(0)JB(0)^* J_*] f, f) = ([J_* - B(0)JB(0)^*] f, f) = \|k_0 f\|^2,$$

and this completes the proof.

4.5. Proposition. *For $F\in h_0\mathfrak{H}_1$, we have*

$$L_3 L_1 B(0) L_2^{-1} F = S^* F. \tag{10}$$

In particular the domain of S^* contains the closure of $h_0\mathfrak{H}_1$, so S^* (and hence S) is bounded.

Proof. We have

$$B(0)L_2^{-1}h_0 f = -B(0)[J-B(0)^*J_*B(0)]^{1/2}f = -[J_*-B(0)JB(0)^*]^{1/2}B(0)f$$

and so

$$L_3L_1B(0)L_2^{-1}h_0 f = -L_3k_0B(0)f = -k_0J_*B(0)f.$$

Thus (10) follows from Proposition 4.3.

Now from Lemma 4.4, the operators L_1, L_2^{-1} and L_3 are bounded, and so the boundedness of S^* on the closure of $h_0\mathfrak{H}_1$ follows from (10). This and Proposition 4.2 show that the domain of S^* contains all of H, and the proposition follows from the Closed Graph Theorem.

5. Characteristic operator function of S^*

5.1. Proposition. *For* $f\in\mathfrak{H}_2$,

$$(I-S^*S)k_0f = k_0[I-B(0)B(0)^*]f, \quad |I-S^*S|k_0f = k_0[J_*-B(0)JB(0)^*]f,$$
$$\operatorname{sgn}(I-S^*S)k_0f = k_0J_*f, \quad |I-S^*S|^{1/2}k_0f = k_0[J_*-B(0)JB(0)^*]^{1/2}f.$$

Proof. The first relation is immediate, since

$$(I-S^*S)k_0f = k_0f+S^*h_0JB(0)^*f = k_0f-k_0J_*B(0)JB(0)^*f = k_0[I-B(0)B(0)^*]f,$$

by Proposition 4.3.

Lemma 4.4 shows L_3 is isometric, and a slight modification of its proof shows L_3 is self adjoint. The above computation shows $L_3(I-S^*S)\geqq 0$ on $k_0\mathfrak{H}_2$ and this proves the second and third relations. The last relation follows from the fact that the operator on the right is positive and its square is $|I-S^*S|$.

5.2. Proposition. *For* $f\in\mathfrak{H}_1$,

$$(I-SS^*)h_0f = h_0[I-B(0)^*B(0)]f, \quad |I-SS^*|h_0f = h_0[J-B(0)^*J_*B(0)]f,$$
$$\operatorname{sgn}(I-SS^*)h_0f = h_0Jf, \quad |I-SS^*|^{1/2}h_0f = h_0[J-B(0)^*J_*B(0)]^{1/2}f.$$

For $0\neq z\in D\cup\tilde{D}$ *and* $f\in\mathfrak{H}_2$,

$$|I-SS^*|^{1/2}k_zf = \bar{z}^{-1}h_0[J-B(0)^*J_*B(0)]^{-1/2}[B(z)^*-B(0)^*]f.$$

Proof. The first four relations follow from a computation similar to the proof of Proposition 5.1, which will be omitted. The last relation follows from Proposition 4.2 and the fact that

$$|I-SS^*|^{1/2}k_zf = |I-SS^*|^{1/2}[\operatorname{sgn}(I-SS^*)](I-SS^*)k_zf.$$

One more lemma, and we shall be able to compute the characteristic operator function of S.

5.3. Lemma. *For* $\bar{z}\in D\cup\tilde{D}$, *we have* $(I-\bar{z}S^*)^{-1}k_0f=k_zf$.

Proof. $(I-\bar{z}S^*)k_zf=k_zf-\bar{z}\bar{z}^{-1}[k_z-k_0f]=k_0f$.

5.4. Theorem. *Up to a constant, isometric multiple, we have* $\Theta_{S*}(z)=B(z)$.

Proof. We find $\Theta_S(z)$ easier to compute. The theorem will follow from a result of Kužel' [4] which states $\Theta_{S*}(z)=\Theta_S(\bar{z})^*$.

We need to know what the range of $I-S^*S$ is, and a computation copied from Proposition 4.2 (using Proposition 4.3) shows that the closures of $R(I-S^*S)$ and $k_0\mathfrak{H}_2$ coincide. What we want is therefore

$$\Theta_S(z)k_0f = S\operatorname{sgn}(I-S^*S)k_0f-z|I-SS^*|^{1/2}(I-zS^*)^{-1}|I-S^*S|^{1/2}k_0f.$$

We first use Proposition 5.1 to get

$$\Theta_S(z)k_0f = Sk_0J_*f-z|I-SS^*|^{1/2}(I-zS^*)^{-1}k_0[J_*-B(0)JB(0)^*]^{1/2}f.$$

Now Lemma 5.3 gives

$$\Theta_S(z)k_0f = Sk_0J_*f-z|I-SS^*|^{1/2}k_{\bar{z}}[J_*-B(0)JB(0)^*]^{1/2}f =$$
$$= Sk_0J_*f-h_0|J-B(0)^*J_*B(0)|^{-1/2}[B(\bar{z})^*-B(0)^*][J_*-B(0)JB(0)^*]^{1/2}f$$

by Proposition 5.2. Now we have

$$h_0|J-B(0)^*J_*B(0)|^{-1/2}B(0)^*[J_*-B(0)JB(0)^*]^{1/2}f = h_0B(0)^*f = -Sk_0J_*f,$$

and so our estimate of Θ_s becomes

$$\Theta_S(z)k_0f = -h_0|J-B(0)^*J_*B(0)|^{-1/2}B(\bar{z})^*[J_*-B(0)JB(0)^*]^{1/2}f = L_2B(\bar{z})^*L_1k_0f$$

and since L_1 and L_2 are (constant) isometries, it follows that $\Theta_S(z)$ and $B(\bar{z})^*$ are the same, considered as characteristic functions (see Section 6 below).

6. Appendix

6.1. Uniqueness of characteristic function. Let S and T be invertible operators on Hilbert spaces H_1 and H_2, and assume S and T have no reducing subspaces on which they are unitary.

Theorem. *If there are constant unitaries U and V such that $\Theta_S=U\Theta_TV$, then S and T are unitarily equivalent.*

Proof. By the computation of Kužel', which we used in Section 1.4, we have

$$(1-z\bar{w})^{-1}[J_*-\Theta_T(z)J\Theta_T(w)^*] = Q_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_* \tag{11}$$

for z and w in $D\cup\tilde{D}$, for D some neighborhood of 0 (where $(I-\bar{z}T)^{-1}$ and $(I-\bar{z}S)^{-1}$

exist). Now let K, K_*, P, P_* denote the analogues of the operators J, J_*, Q, Q_* corresponding to S. We have $K=V^*JV$, $K_*=UJ_*U^*$, and so

$$K_*-\Theta_S(z)K\Theta_S(w)^* = U[J_*-\Theta_T J\Theta_T^*]U^*.$$

Combining this with (11), we have

$$P_*(I-zS^*)^{-1}(I-\bar{w}S)^{-1}P_* = UQ_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_*U^*.$$

For $g, h\in\bar{R}(P_*)$, we therefore have

$$((I-\bar{w}S)^{-1}P_*g, (I-\bar{z}S)^{-1}P_*h) = ((I-\bar{w}T)^{-1}Q_*U^*g, (I-\bar{z}T)^{-1}Q_*U^*h)$$

whence it follows that the map

$$\mathfrak{U}: (I-\bar{w}S)^{-1}P_*g \mapsto (I-\bar{w}T)^{-1}Q_*U^*g, \quad w\in D\cup\tilde{D}, \tag{12}$$

is an isometry, from some subspace of H_1 to some subspace of H_2. If we can prove that the subspace M of H_1 of elements of the form $f(S)P_*g$, where $g\in\bar{R}(P_*)$, and f is a rational function with poles in $D\cup\tilde{D}$, is dense in H_1, then we will have that $\mathfrak{U}$ has a dense domain.

To prove M is dense in H_1, we shall prove that $\bar{M}$ reduces S and that S is unitary on $M^\perp$. M is certainly S invariant. To prove M is S^* invariant, note

$$S^*f(S)P_*g = S^{-1}[(SS^*-I)f(S)P_*g]+S^{-1}f(S)P_*g.$$

Let $h = |I-SS^*|^{1/2}K_*f(S)P_*g$, and we have

$$S^*f(S)P_*g = -S^{-1}P_*h+S^{-1}f(S)P_*g\in M.$$

Now we want to show that M contains the range of $I-S^*S$, i.e. $(I-S^*S)H_1$. To do this, let $f=(I-S^*S)g$. We have

$$f = S^{-1}(I-SS^*)Sg = S^{-1}P_*h \quad \text{where} \quad h = P_*KSg\in R(P_*).$$

Now M reduces S and contains $R(P)$ and $R(P_*)$, so S must be unitary on $M^\perp$; i.e. $M^\perp=\{0\}$. A similar argument shows the range of $\mathfrak{U}$ is dense in H_2, so $\mathfrak{U}$ is unitary.

To complete the proof of the theorem, we must show $S=\mathfrak{U}^*T\mathfrak{U}$. To do this, just refer back to (12). It implies

$$\mathfrak{U}f(S)P_*g = f(T)Q_*U^*g.$$

Replacing $f(t)$ by $tf(t)$ here, we get

$$\mathfrak{U}Sf(S)P_*g = Tf(T)Q_*U^*g = T\mathfrak{U}f(S)P_*g,$$

so that $\mathfrak{U}Sx=T\mathfrak{U}x$ for $x\in M$, and this completes the proof.

A corollary of the theorem is that any bounded, invertible operator is unitarily equivalent to the operator S constructed in Section 3.3.

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