On the nilstufe of homogeneous groups

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1. All groups in this paper are abelian, with addition the group operation. A homogeneous group is a group all of whose elements are of the same type (see [1, p. 147] for a definition of type). We define the type of a homogeneous group to be the type of any of its non-zero elements. For G a group, and $g\in G$, the type of g will be denoted by $T(g)$, and if G is homogeneous, the type of G will be denoted by $T(G)$.

SZELE [3] defined the nilstufe of a group G to be n, n a positive integer, if there exists an associative ring R with additive group G such that $R^n\neq 0$, but for every associative ring R with additive group G holds $R^{n+1}=0$. If no such positive integer n exists, we will say that G has nilstufe ∞. By considering not necessarily associative rings with additive group G, we may similarly define the strong nilstufe of G. The nilstufe of G will be denoted by $\nu(G)$ and the strong nilstufe by $N(G)$. A multiplication on a group G is meant to be the multiplication of a ring R with additive group G.

RÉDEI and SZELE [2] have shown that if G is a rank one, torsion-free group, and if the components of $T(G)$ are not all 0 and ∞, then $\nu(G)=1$. We will show more generally (corollary to theorem 1) that if G is a homogeneous group, and if the components of $T(G)$ are not all 0 and ∞, then $N(G)=1$. Under certain restrictions, the nilstufe of a direct sum of homogeneous groups will be computed.

2. Lemma. *Let G, H, and K be homogeneous groups. If $T(K)\not\geq T(G)+T(H)$ then* $\mathrm{Hom}\,(G\otimes H, K)=0$.

Proof. Let $0\neq g\in G$, $0\neq h\in H$, and let $\varphi\in\mathrm{Hom}\,(G\otimes H, K)$. If $\varphi(g\otimes h)\neq 0$, then $T(K)=T[\varphi(g\otimes h)]\geq T(g\otimes h)\geq T(g)+T(h)=T(G)+T(H)$, a contradiction. Therefore $\varphi=0$, and $\mathrm{Hom}\,(G\otimes H, K)=0$.

Theorem 1. *Let $\{G_i, i\in I\}$ be a set of homogeneous groups and let $G=\sum_{i\in I}^{n}\oplus G_i$. If for all $i, j, k\in I$, $T(G_k)\not\geq T(G_i)+T(G_j)$, then $N(G)=1$.*

Proof. Let Mult G be the group of multiplications on G. Then, by the lemma,

$$\text{Mult}\, G \simeq \text{Hom}\,(G\otimes G, G) = \text{Hom}\,(\sum_{i,j\in I} \oplus G_i\otimes G_j, \sum_{k\in I}\oplus G_k) \subseteqq$$
$$\subseteqq \sum_{i,j,k\in I} \oplus \text{Hom}\,(G_i\otimes G_j, G_k) = 0.$$

Corollary. *Let G be a homogeneous group. If the components of $T(G)$ are not all 0 and ∞, then $N(G)=1$.*

Theorem 2. *Let G_i be a homogeneous group, with the components of $T(G_i)$ not all 0 and ∞, $1\leqq i\leqq n$. Let $G=\sum_{i=1}^{n}\oplus G_i$. If for $i\neq j$, $T(G_i)\neq T(G_j)$, $1\leqq i,j\leqq n$, then*

$$\nu(G) \leqq 2^n-1.$$

Proof. For $n=1$ the theorem is true by theorem 1. Let $n>1$ and suppose the theorem is true for the direct sum of $n-1$ homogeneous groups. It may be assumed that G_n is such that $T(G_i)\not\geqq T(G_n)$ for $1\leqq i\leqq n-1$. Let $g\in G_n$, and let $h\in G$. For any multiplication on G, $T(gh)\geqq T(g)\geqq T(G_n)$. However $gh=\sum_{i=1}^{n} g_i$, $g_i\in G_i$, $1\leqq i\leqq n$. For $1\leqq i\leqq n-1$ if $g_i\neq 0$, then $T(G_i)=T(g_i)\geqq T(gh)\geqq T(G_n)$, a contradiction. Therefore:

(1) $$G_nG \subseteqq G_n,$$

and similarly

(2) $$GG_n \subseteqq G_n.$$

Let $g, g'\in G_n$. For any multiplication on G, $T(gg')>T(G_n)$. However, $gg'=\sum_{i=1}^{n} g_i$. If $g_i\neq 0$, $1\leqq i\leqq n$, then $T(G_i)=T(g_i)\geqq T(gg')>T(G_n)$, a contradiction. Therefore

(3) $$G_n^2 = 0.$$

(1) and (2) imply that G_n is an ideal in every ring on G. Therefore, every multiplication on G induces a multiplication on G/G_n. By the induction hypothesis $(G/G_n)^{2^{n-1}}=0$. Hence:

(4) $$G^{2^{n-1}} \subseteqq G_n.$$

(3) and (4) yield that $G^{2^n}=(G^{2^{n-1}})^2\subseteqq G_n^2=0$, and hence $\nu(G)\leqq 2^n-1$.

References

[1] L. Fuchs, *Abelian Groups*, Akadémiai Kiadó (Budapest, 1966).

[2] L. Rédei—T. Szele, Die Ringe ersten Ranges, *Acta Sci. Math.*, **12A** (1950), 18—29.

[3] T. Szele, Gruppentheoretische Beziehungen bei gewissen Ringkonstruktionen, *Math. Z.*, **54** (1951), 168—180.

(Received March 21, 1973)