The nilstufe of rank two torsion free groups

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1. It is well known [5] that a ring R whose additive group is a rank one torsion free group is either a zero-ring ($xy=0$ for all $x, y \in R$) or isomorphic to a subring of the field of rational numbers.

SZELE [6] introduced the notion of the nilstufe of an abelian group G. (In what follows a group is always meant to be an abelian group with addition the group operation.) Let n be a positive integer. The nilstufe of G is said to be n, denoted $N(G)=n$, if there exists a multiplication on G, not necessarily associative, such that $G^n \neq 0$, but $G^{n+1}=0$ under every multiplication on G. If, for every positive integer n, there exists a multiplication on G such that $G^n \neq 0$, then G is said to have nilstufe ∞, denoted $N(G)=\infty$. An immediate consequence of the result mentioned in the previous paragraph is that if G is a rank one torsion free group, then $N(G)=1$ or ∞. The objective of this paper is to show that if G is a rank two torsion free group, then $N(G)=1, 2$ or ∞. One is naturally led to conjecture that if G is a rank n torsion free group, then $N(G)=1, 2, \ldots, n$, or ∞.

The major tools used to compute $N(G)$ are results of BEAUMONT and WISNER [1] concerning multiplications on a rank two torsion free group. These results are introduced in section 2. In section 3, $N(G)$ is computed for G a rank two torsion free group. Sufficient conditions are given for G to be a nil-group (i.e., $N(G)=1$) if G is the direct sum of rank one torsion free groups in section 4. Quasi-equality and quasi-decomposability are discussed in section 5, and their effect on the nilstufe is considered.

2. Definition 1. If G is the additive group of a ring R, then R is called a ring over G.

Lemma 1. *Let G be a rank two torsion free group, and let R be a ring over G. If R is non-commutative, then x and x^2 are dependent for all $x \in G$. If $R \neq 0$, and R is commutative, then there exists an $x \in G$ such that x and x^2 are independent.*

Proof: [1, p. 108].

Lemma 2. *Let G be a rank 2 torsion free group, R a non-commutative ring over G. Then there exists an $x \in G$ such that $x^2 \neq 0$.*

Proof: [1, p. 109, lemma 2].

3. Theorem 1. *Let G be a rank two torsion free group. Then $N(G)=1$, 2 or ∞.*

Proof. Suppose there exists a non-commutative ring R over G. By lemma 2 there exists an $x \in G$ such that $x^2 \neq 0$. x and x^2 are dependent by lemma 1, therefore there exist non-zero integers n, m such that $nx^2 = mx$. Suppose $x^k \neq 0$ for some positive integer k. $nx^{k+1} = mx^k \neq 0$, since G is torsion free. Therefore $x^{k+1} \neq 0$, and we have shown inductively that $x^n \neq 0$ for all positive integers n, hence $N(G)=\infty$.

It therefore suffices to consider the case G a group over which there are no non-commutative rings. Let G be such a group and suppose $N(G)=n$, $2<n<\infty$. Let R be a ring over G, $R \neq 0$. R is commutative, so that by lemma 1 there is an $x \in G$ such that x and x^2 are independent.

Let $g_1, \ldots, g_n \in G$. There exist integers k_i, l_i, m_i ($k_i \neq 0$), such that $k_i g_i = l_i x + m_i x^2$ for $1 \leqq i \leqq n$. $N(G)=n$; therefore

$$\left(\prod_{i=1}^{n} k_i\right)\left(\prod_{i=1}^{n} g_i\right) = \left(\prod_{i=1}^{n} l_i\right) x^n.$$

If we can show that $x^n = 0$, then by virtue of the fact that $\prod_{i=1}^{n} k_i \neq 0$, and that G is torsion free, we will have that $\prod_{i=1}^{n} g_i = 0$, thus contradicting the fact that $N(G)=n$. There exist integers k, l, m, $k \neq 0$ such that $kx^n = lx + mx^2$. $N(G)=n$ therefore $0 = kx^{n+1} = lx^2 + mx^3$. If $l=0$ or $m=0$, then $x^3=0$. Since $n>2$, we have that $x^n=0$. If $l \neq 0$ and $m \neq 0$, then $lx^n = x^{n-2}(lx^2) = x^{n-2}(-mx^3) = -mx^{n+1} = 0$. G is torsion free, therefore $x^n = 0$.

Corollary 1. *Let G be a rank two torsion free ring. If there exists a non-commutative ring R over G, then $N(G)=\infty$.*

4. Let H be a rank one torsion free group. All non-zero elements of H have the same type. We therefore denote by $T(H)$ the type of any non-zero element of H, and call $T(H)$ the type of H.

Lemma 3. *Let G_1 and G_2 be rank one torsion free groups, then $G_1 \otimes G_2$ is a rank one torsion free group, and $T(G_1 \otimes G_2) = T(G_1) + T(G_2)$.*

Proof. [4, p. 255 and p. 261].

Lemma 4. *Let H and K be rank one torsion free groups. If $T(H) \not\leqq T(K)$, then* Hom $(H, K)=0$.

Proof. Let $h \in H$, $h \neq 0$, and $\varphi \in \mathrm{Hom}(H, K)$. Put $k = \varphi(h) \neq 0$. By $T(H) \not\leqq T(K)$, there exist a prime p and a positive integer l such that $p^l | h$, $p^l \nmid k$. Hence $h = p^l h'$, $h' \in H$. Therefore $k = \varphi(h) = p^l \varphi(h')$, a contradiction.

Theorem 2. *Let $G = G_1 \oplus G_2$, G_1 and G_2 rank one torsion free groups. If $2T(G_1) \not\leqq T(G_2)$ and $2T(G_2) \not\leqq T(G_1)$, then $N(G) = 1$.*

Proof. Let Mult G be the group of multiplications which can be defined on G; $\mathrm{Mult}\, G \cong \mathrm{Hom}(G \otimes G, G) = \sum_{i,j,k=1}^{2} \mathrm{Hom}(G_i \otimes G_j, G_k) = 0$ by lemmata 3 and 4.

5. Definition 2. A group G is said to be *quasi-contained* in a group H, denoted $G \dot{\subset} H$, if there exists an integer $n \neq 0$ such that $nG \subset H$. If $G \dot{\subset} H$ and $H \dot{\subset} G$, then G is said to be quasi-equal to H, $G \doteq H$.

Theorem 3. *Let G and H be torsion free groups. If $G \doteq H$, then $N(G) = N(H)$.*

Proof. $G \doteq H$; therefore there exist non-zero integers k, l, $kG \subset H$, and $lH \subset G$. Suppose $N(G) = \infty$. Let n be an arbitrary positive integer. There exist a multiplication $\times_G$ on G and elements $g_i \in G$ $(1 \leqq i \leqq n)$, such that $g_1 \times_G g_2 \times_G \dots \times_G g_n \neq 0$. Let $h_1, h_2 \in H$. Define $h_1 \times_H h_2 = (lh_1) \times_G (lh_2)$. $\times_H$ is a multiplication on H. $(kg_1) \times_H \cdot (kg_2) \times_H \dots \times_H (kg_n) = (k^n l^n) g_1 \times_G g_2 \times_G \dots \times_G g_n \neq 0$ since G is torsion free. Therefore $N(H) = \infty$. Similarly, we may prove that if $N(H) = \infty$ then $N(G) = \infty$. We may therefore assume that $N(G)$ and $N(H)$ are both finite.

Let $N(G) = n$, and let $\times_H$ be a multiplication on H. Let $g_1, g_2 \in G$. Define $g_1 \times_G g_2 = (kg_1) \times_H (kg_2)$. $\times_G$ is a multiplication on G. Let $h_1, \dots, h_n, h_{n+1} \in H$.

$$(k^{n+1} l^{n+1})(h_1 \times_H h_2 \times_H \dots \times_H h_n \times_H h_{n+1}) =$$

$$= (klh_1) \times_H (klh_2) \times_H \dots \times_H (klh_n) \times_H (klh_{n+1}) =$$

$$= (lh_1) \times_G (lh_2) \times_G \dots \times_G (lh_n) \times_G (lh_{n+1}) = 0.$$

H is torsion free, therefore $h_1 \times_H \dots \times_H h_n \times_H h_{n+1} = 0$, so that $N(H) \leqq n = N(G)$. Similarly we can prove that $N(G) \leqq N(H)$, so that $N(G) = N(H)$.

Definition 4. A group G is said to be *quasi-decomposable* if there exist non-zero groups A, B such that $G \doteq A \oplus B$.

Corollary 2. *Let G be a quasi-decomposable rank two torsion free group, $G \doteq G_1 \oplus G_2$. If $2T(G_1) \not\leqq T(G_2)$ and $2T(G_2) \not\leqq T(G_1)$ then $N(G) = 1$.*

Proof. By theorem 2, $N(G_1 \oplus G_2) = 1$, and by theorem 3, $N(G) = N(G_1 \oplus G_2)$.

References

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