On operator radii

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For a bounded linear operator T on a (real or complex) Banach space X, one has the relation

$$|\sigma(T)| \leqq |W(T)| \leqq \|T\|$$

between the spectral radius $|\sigma(T)|$, the numerical radius $|W(T)|$, and the operator radius $\|T\|$ (see definitions below). In a complex Banach space one has additionally that

$$\|T\| \leqq c|W(T)|,$$

where $c=2$ for a complex Hilbert space X (e.g., see [6]), whereas $c=e$ (see [1], [4], [9]) for a complex Banach space.

In this note we will examine the relations between these three radii $|\sigma(T)|$, $|W(T)|$, and $\|T\|$ for an arbitrary densely defined operator T in X.

We recall the definitions:

$$|\sigma(T)| = \sup|\lambda|, \quad \lambda \text{ in the spectrum } \sigma(T),$$

$$|W(T)| = \sup|\lambda|, \quad \lambda \text{ in the numerical range } W(T),$$

$$\|T\| = \sup\|Tx\|, \quad x \text{ in the domain } D(T) \text{ of } T, \quad \|x\| = 1,$$

where $W(T)=\{x^* Tx \mid x\in D(T),\ \|x\|=1,\ x^*\in J(x)\}$ and

$$J(x) = \{x^*\in X^* \mid x^*x = \|x\|^2 = \|x^*\|^2\}.$$

$J(x)$ denotes the totality of the "Hahn—Banach" duality vectors $x^*\in X^*$ for a given x, whereas here the numerical range $W(T)$ is to be understood as defined in terms of a single x^* selected from $J(x)$ for each x. Sometimes (e.g., see [2]) the numerical range of T is defined by $V(T)=\{x^* Tx \mid x\in D(T),\ \|x\|=1, \text{ all } x^*\in J(x)\}$, i.e. $V(T)= =\bigcup W_\varphi(T)$, for all functions $\varphi: D(T)\to J(D(T))$. Each such function $\varphi: X\to J(X)$ defines a semi-inner product $[y, x]=x^* y$ on X, and conversely each semi-inner product consistent with the norm $\|x\|$ is given exactly by a φ. For further information con-

* Partially supported by NSF GP 15239 A-1.

cerning semi-inner products and numerical ranges for bounded operators and Banach-algebras see the recent book by BONSALL—DUNCAN [2].

The general situation for the four cases 1) X a real Hilbert space, 2) X a complex Hilbert space, 3) X a real Banach space, 4) X a complex Banach space, is summarized by the following theorem.

Theorem. *Let T be a densely defined linear operator in X; then in cases* 1), 3), 4)

$$|\sigma(T)| = \infty \Rightarrow \|T\| = \infty \Leftarrow |W(T)| = \infty$$

and in case 2)

$$|\sigma(T)| = \infty \Rightarrow \|T\| = \infty \Leftrightarrow |W(T)| = \infty.$$

In 1), 2), *and* 3) *all other implications are false in general. In* 4), *for T closed, $\|T\| = \infty$ implies that $|W(T)| = \infty$ or $|\sigma(T)| = \infty$.*

Proof. We will consider in turn the six possible implications between the three conditions

$$|\sigma(T)| = \infty, \quad \|T\| = \infty, \quad |W(T)| = \infty.$$

In all cases $|\sigma(T)| = \infty \Rightarrow \|T\| = \infty$ follows from the defect index theory, and $|W(T)| = \infty \Rightarrow \|T\| = \infty$ follows from the Schwarz inequality.

The possible implication $\|T\| = \infty \Rightarrow |\sigma(T)| = \infty$ in case 1) (and hence case 3)) is ruled out by the example $T_1 = \oplus \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}$ $(n = 1, 2, 3, \ldots)$, the direct sum operator in real $l_R^2 = X$ with $D(T_1) = M$, the subspace of l^2 consisting of all vectors which have only a finite number of nonzero components. T_1 is unbounded, $\sigma(T_1)$ is empty and $W(T_1) = \{0\}$. To obtain a closed counterexample, one may observe that the closed operator $T_2 = \hat{T}_1$, the closure of T_1, has the same properties. The derivative operator $T_3 u = u'$, $D(T_3) = \{u \mid u$ absolutely continous, $u' \in L^2$, $u(0) = 0\} \subset L_C^2(0, 1) = X$ has empty spectrum and is closed and unbounded, and hence serves to negate this implications also in the cases 2) and 4).

The implication $|W(T)| = \infty \Rightarrow |\sigma(T)| = \infty$ is ruled out for the cases 1) and 3) by the example $T_4 = \oplus \begin{pmatrix} 0 & n \\ -n^2 & 0 \end{pmatrix}$ $(n = 1, 2, 3, \ldots)$ in l_R^2 with $D(T_4) = M$, since $W(T_4)$ is unbounded but $\sigma(T_4)$ is empty. For a closed counterexample with the same properties as T_4, take $T_5 = \hat{T}_4$. The counterexample T_3 given above negates the complex cases 2) and 4), since $|W(T_3)| = \infty$.

The remaining two possible implications are

$$\|T\| = \infty \Rightarrow |W(T)| = \infty \quad \text{and} \quad |\sigma(T)| = \infty \Rightarrow |W(T)| = \infty.$$

The example T_2 rules out the first implication in the cases 1) and 3), since $W(T_2) = \{0\}$, and the following example T_6 negates both implications in the cases 1) and 3). Let $T_6 u = u'$ with $D(T_6) = \{u \mid u$ absolutely continuous, $u' \in L^2$, $u(0) =$

$=u(1)=0\}\subset L_R^2(0,1)=X$. Then T_6 is unbounded, but $W(T_6)=\{0\}$ because for $u\in D(T_6)$ one has $(T_6u,u)=\frac{1}{2}\int\limits_0^1[u^2(x)]'\,dx=0$; moreover $|\sigma(T_6)|=\infty$ because the residual spectrum $\sigma_r(T_6)$ is the whole real line (since $R(\lambda-T_6)\perp e^{\lambda x}$ for each real λ).

In the case 2) of X a complex Hilbert space both of the above mentioned remaining two implications are true. It suffices of course to demonstrate the first (perhaps known). Let T be unbounded and densely defined and suppose that $|W(T)|<\infty$. Then by polarization and the parallelogram law, one has for $x,y\in D(T)$, that

$$|(Tx,y)|\leqq|W(T)|\cdot 4^{-1}[\|x+y\|^2+\|x+iy\|^2+\|x-y\|^2+\|x-iy\|^2]=$$
$$=|W(T)|\cdot[\|x\|^2+\|y\|^2],$$

so that $|(Tx,y)|=\|x\|\cdot\|y\|\cdot|(\|x\|^{-1}Tx,\|y\|^{-1}y)|\leqq 2|W(T)|\cdot\|x\|\cdot\|y\|$. Since $D(T)$ is dense, $\|Tx\|\cdot\|x\|^{-1}\leqq 2|W(T)|<\infty$, and T is bounded. Finally, in case 4) of X a complex Banach space and T a closed operator, it is known (KATO [7, p. 176]) that if $|\sigma(T)|<\infty$ then $\|T\|=\infty$ if and only if the resolvent operator $(\lambda-T)^{-1}$ has an essential singularity at infinity. Hence if both $|\sigma(T)|<\infty$ and $|W(T)|<\infty$, by noting that the latter implies that $\|(\lambda-T)^{-1}\|\to 0$ as $|\lambda|\to 0$, one has $\|T\|<\infty$. This concludes the proof of the theorem.

Remarks. We conclude with the following remarks.

1. The implications $|\sigma(T)|=\infty\Rightarrow\|T\|=\infty\Leftarrow|W(T)|=\infty$ clearly hold in a normed linear space also.

2. A special situation arises when T is everywhere defined on a Banach space X, i.e. when $D(T)=X$. By a well-known "metatheorem", then almost any additional condition will make T bounded.*)

In this situation, when $|(W(T)|<\infty$, by the closeability of T (see remark 3 below) one knows that T is closed and hence bounded (by the closed graph theorem).

Moreover, by the following arguments (perhaps known) it follows that $|\sigma(T)|<\infty$ and $D(T)=X$ imply that T is bounded.

a) Let $D(T)=X$; then T^* is bounded. This can be seen by letting $z_n^*=T^*y_n^*$ for any sequence $\{y_n^*\}$ in $D(T^*)$, $\|y_n^*\|=1$; fixing x, one has $z_n^*(x)=T^*y_n^*(x)=$ $=y_n^*Tx\leqq\|Tx\|$ so that (by the uniform boundedness principle) $\{\|T^*y_n^*\|\}$ is a bounded set.

b) Let $|\sigma(T)|<\infty$, $D(T)=X$; then by a) T^* is bounded. For $|\lambda|>\|T^*\|$ one has $0=\operatorname{codim}\overline{R(\lambda-T)^*}=\operatorname{codim}\overline{R(\bar\lambda I|D(T^*))}=\operatorname{codim}\overline{D(T^*)}$, so that $D(T^*)$ is dense, and hence $D(T^*)=X^*$, which holds if and only if T is bounded.

*) For example, this has been recently put on a logical basis by M. AJTAI, On the boundedness of definable linear operators, *Periodica Math. Hungarica* (to appear).

In summary, when $D(T)=X$ a real or complex Banach space, one has

$$|\sigma(T)| = \infty \Leftrightarrow \|T\| = \infty \Leftrightarrow |W(T)| = \infty.$$

3. It is known (see KATO [7, p. 268]) for a Hilbert space that if $W(T)$ is not the whole plane, then T is closeable. This generalizes (e.g., see [10], [11]) to a Banach space when $W(T)$ is in a half plane (or half line in the real case.) Let us observe here that one can say roughly that some $W_\varphi(T)$ "not the whole complex plane" implies that T is closeable in the Banach space also. In particular, this will be the case when $W(T)$ misses an external sector somewhere in the plane; other geometrical situations that are included will be evident from the proof.

More precisely, let there exist a sequence of scalars $\{\lambda_k\}$, $|\lambda_k|\to\infty$, such that $d(\lambda_k, W(T))/|\lambda_k|\not\to 0$, and let T be densely defined in a normed linear space X (X either real or complex); then T is closeable.

Suppose, to the contrary, that there exists a sequence $x_n \in D(T)$, $x_n\to 0$, $Tx_n\to y$, $\|y\|=1$. By hypothesis we may assume $d(\lambda_k, W(T))/|\lambda_k| \geqq \varepsilon > 0$, for some fixed ε. By $D(T)$ dense, there exists $z_\varepsilon \in D(T)$, $\|z_\varepsilon\|=1$, $\|z_\varepsilon - y\| < \varepsilon/2$. Let

$$g(n,k) = \|\lambda_k x_n + z_\varepsilon - y - \lambda_k^{-1} T z_\varepsilon\|;$$

then

$$\lim_{n\to\infty} g(n,k) = \|z_\varepsilon - y - \lambda_k^{-1} T z_\varepsilon\| < \varepsilon/2 + |\lambda_k|^{-1}\|Tz_\varepsilon\|,$$

for fixed k. On the other hand, letting

$$u_{nk} = (x_n + \lambda_k^{-1} z_\varepsilon)\,\|x_n + \lambda_k^{-1} z_\varepsilon\|^{-1},$$

one has by Schwarz's inequality that

$$g(n,k) = \|(\lambda_k - T)(x_n + \lambda_k^{-1} z_\varepsilon) + (Tx_n - y)\| \geqq \|(\lambda_k - T)(x_n + \lambda_k^{-1} z_\varepsilon)\| - \|Tx_n - y\| \geqq$$

$$\geqq |\lambda_k - [Tu_{nk}, u_{nk}]|\,\|x_n + \lambda_k^{-1} z_\varepsilon\| - \|Tx_n - y\| \geqq d(\lambda_k, W(T))\,\|x_n + \lambda_k^{-1} z_\varepsilon\| - \|Tx_n - y\|.$$

Hence

$$\lim_{n\to\infty} g(n,k) \geqq d(\lambda_k, W(T))/|\lambda_k| \geqq \varepsilon.$$

But from the first estimate above, noting that $\|Tz_\varepsilon\|$ does not depend on k, one has $\varepsilon > \lim g(n,k)$ for k sufficiently large, contradicting the second estimate.

We mention that for X such that J is single valued and continuous (e.g., see [1], [3], [8]), one has additionally for closeable T that $\overline{W(\hat{T})} = \overline{W(T)}$ as in the Hilbert space case, since $x_n \to x$, $Tx_n \to \hat{T}x$ imply that $x_n{}^* Tx_n \to x^* \hat{T}x$.

4. Although we have not done so here, one can make $|\sigma_{ext}(T)|=\infty \Leftrightarrow \|T\|=\infty$ by using the notion of extended spectrum (e.g., see [7]).

5. Of course, not all of the considered implications are independent. In particular, one has $\{\|T\|=\infty \Rightarrow |W(T)|=\infty\} \Leftrightarrow \{|\sigma(T)|=\infty \Rightarrow |W(T)|=\infty\}$ in case 4): to the right, by the previously noted general implications; and to the left, by the following argument. Given $\|T\|=\infty$, if $|W(T)|<\infty$, then by the right hand implication we would have $|\sigma(T)|<\infty$, and then, using the result [7, p. 176] already used above, one has $\|T\|<\infty$, a contradiction.

6. To recapitulate, exactly the following situations occur:

a) $\|T\| < \infty,\quad |\sigma(T)| < \infty,\quad |W(T)| < \infty$ cases 1)—4)

b) $\|T\| = \infty,\quad |\sigma(T)| = \infty,\quad |W(T)| = \infty$ cases 1)—4)

c) $\|T\| = \infty,\quad |\sigma(T)| < \infty,\quad |W(T)| = \infty$ cases 1)—4)

d) $\|T\| = \infty,\quad |\sigma(T)| = \infty,\quad |W(T)| < \infty$ cases 1), 3), not 2)

e) $\|T\| = \infty,\quad |\sigma(T)| < \infty,\quad |W(T)| < \infty$ cases 1), 3), not 2), not 4) for T closed.

7. There remains the question of whether $\|T\|=\infty \Rightarrow |W(T)|=\infty$ in the case 4). An exception to this clearly cannot occur, for example, when any of the following conditions prevails: a) $|\sigma(T)|<\infty$; b) $\exists\lambda\in\varrho(T),\ |\lambda|>|W(T)|$; c) $|W(T^*)|<\infty$; d) $J(D(T))$ contains an eigenvector of $(\bar{\lambda}-T^*)$, $|\lambda|>W(T)$.

8. Finally we mention that one can construct a proof in the case 2) different from that given above; this proof completely avoids both polarization and the parallelogram law but still requires a bilinear form. The argument is similar to that used in [5] to show that the cosine of an unbounded operator is always zero, and we omit the details.

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(Received December 15, 1972, revised October 15, 1973)