Normal extensions of subnormal operators

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1. Introduction. Only bounded operators on Hilbert spaces will be considered below. Let T be subnormal on $\mathfrak{H}$ and let N on $\mathfrak{K} \supset \mathfrak{H}$ denote the minimal normal extension of T. (Concerning subnormal operators and their basic properties, see HALMOS [6], pp. 100 ff.) It was shown by HALMOS [5] that sp (N) is a subset of sp (T) and by BRAM [1] that, in fact, sp (T) consists of sp (N) together with some of the holes of sp (N); cf. [6], p. 102. A subnormal T will be called completely subnormal if there exists no non-trivial reducing space on which it is normal.

It is known that if T is isometric ($T^*T=I$) then T is subnormal and if, in addition, T is completely subnormal, that it is the direct sum of (one or more) copies of the unilateral shift; cf. [6], p. 74. Since the bilateral shift is the minimal normal (here, unitary) extension of the unilateral shift, the minimal unitary extension of a completely subnormal isometry is the direct sum of bilateral shifts.

If A is self-adjoint on a Hilbert space with the spectral resolution $A=\int t\,dE_t$, then the set $\mathfrak{H}_a(A)$ of vectors x for which $\|E_t x\|^2$ is an absolutely continuous function of t is a reducing space of A. A similar statement holds for a unitary operator $U=\int_0^{2\pi} e^{it}\,dE_t$. (The usual assumptions are made here, namely, that E_t is continuous from the right and that, in the unitary case, $E_0=0$, hence E_t is continuous at $t=0$, and $E_{2\pi}=I$.) The operator A or U is said to be absolutely continuous if $\mathfrak{H}_a(A)$ or $\mathfrak{H}_a(U)$ is the entire Hilbert space.

It is well-known that the bilateral shift is absolutely continuous with spectrum $\{z:|z|=1\}$; for a proof using commutators, see PUTNAM [9], p. 22. It follows that the minimal unitary extension of a completely subnormal isometry has the same properties, a result which will be generalized below. Some preliminaries will first be needed.

Let N be a normal operator on a Hilbert space $\mathfrak{K}$ with the spectral resolution

$$N=\int z\,dK. \tag{1.1}$$

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For each subset A of the complex plane, $\mathbf{C}$, let $p(A)$ denote the "radial projection" of A into the circle $|z|=1$ defined by $p(A)=\{p(z): z\in A\}$, where $p(0)=1$ and $p(z)=e^{it}$ if $z\neq 0$ and $z=|z|e^{it}$. Call N radially absolutely continuous if $K(A)=0$ whenever A is a planar Borel set whose radial projection $p(A)$ has measure 0 on $|z|=1$, the measure being ordinary Lebesgue arc length. Let U denote the unitary operator defined by

$$(1.2)\qquad U=\int_0^{2\pi} e^{it}\,dE_t, \quad \text{where} \quad E_t=K(A_t),$$

with $A_t=\{z: z\neq 0,\ 0<\arg z\leqq t\}$ for $0<t<2\pi$ and $A_{2\pi}=\mathbf{C}$ (and $E_t=0$ or $E_t=I$ according as $t\leqq 0$ or $t>2\pi$). Then, to say that N is radially absolutely continuous means that U is absolutely continuous as defined earlier.

Theorem 1. *Let T be a completely subnormal operator on a Hilbert space $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K}$ and let Q denote the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Suppose that*

$$(1.3)\qquad Q(N^*N)=(N^*N)Q.$$

Then N is radially absolutely continuous, that is, U defined by (1.1) *and* (1.2) *is absolutely continuous. Further,*

$$(1.4)\qquad \operatorname{sp}(U)=\{z:|z|=1\}.$$

It is seen that if N is normal on $\mathfrak{K}$ with spectral resolution (1.1) then N has a polar representation $N=UP(=PU)$, where

$$(1.5)\qquad P=(N^*N)^{1/2}$$

and U is defined by (1.2). If (1.3) holds, that is, if $QP^2=P^2Q$, then, since $P\geqq 0$, $QP=PQ$, so that $\mathfrak{H}$ is invariant under P.

If N is unitary, then (1.3) holds trivially. Further, $P=I$ and $N=UP=U$. Thus, it follows from Theorem 1 that $N(=U)$ is absolutely continuous and that its spectrum is the entire circle $|z|=1$. In fact, as previously noted, much more is known: U is a direct sum of bilateral shifts. That the minimal normal extension N of a completely subnormal T may fail to be radially absolutely continuous if (1.3) is not assumed is easy to show by examples; cf. section 4 below. Further, if (1.3) fails to hold for T, it may be possible to replace T by another completely subnormal operator T_1 on a Hilbert space $\mathfrak{H}_1$, in such a way that the minimal normal extension of T_1 is a part, N_1, of N and such that $\mathfrak{H}_1$ is invariant under N_1 and $N_1^*N_1$. Then (1.3) would hold with Q replaced by Q_1, the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}_1$. See the example in section 4 below.

Roughly speakly, condition (1.4) says that the spectrum of N surrounds the origin. More precisely, relation (1.4) holds if and only if there does not exist an open wedge

$$W = \{z : z = re^{it},\ r > 0,\ a < t < b\}, \tag{1.6}$$

for which

$$\operatorname{sp}(N) \cap W \text{ is empty.} \tag{1.7}$$

This fact is easily deduced from the definition (1.2) of U. Note that (1.7) may hold even though 0 is in the spectrum of N, although (1.7) does imply, of course, that 0 cannot be an interior point of $\operatorname{sp}(N)$.

Theorem 2. *Let T be completely subnormal on $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K}$. Suppose that there exists some wedge W of* (1.6) *satisfying* (1.7). *Then $\mathfrak{K}$ is the least space containing $\mathfrak{H}$ and invariant under N and N^*N.*

The proof of Theorem 1 will be given in section 2 and will depend on certain results on commutators; see [9], pp. 21—22. Theorem 2 will be proved in section 3 as a consequence of Theorem 1. Examples illustrating Theorems 1 and 2 will be given in sections 4 and 5. In particular, Theorem 3 of section 5 is an application of Theorem 2 to certain analytic position operators. Finally, some remarks relating absolute continuity of normal operators and second order commutators will be made in section 6.

2. Proof of Theorem 1. Since T is subnormal, it is also hyponormal and hence

$$T^*T - TT^* = D, \quad \text{where} \quad D \geqq 0. \tag{2.1}$$

Further, for $x \in \mathfrak{H}$, one has $T^*x = QN^*x$ (cf. [6], p. 103), thus $QN^*Nx = NQN^*x = Dx$ for $x \in \mathfrak{H}$. Let now the corresponding equation be considered on $\mathfrak{K}$, so that

$$QN^*N - NQN^* = D_1, \tag{2.2}$$

with $D_1 x = Dx$ for x in $\mathfrak{H}$. In view of (1.3) it is seen that D_1 is self-adjoint. Further, $\mathfrak{H}$ (hence $\mathfrak{H}^\perp$) is invariant under D_1 and clearly

$$D_1 = D \oplus 0 \quad \text{on} \quad \mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}^\perp. \tag{2.3}$$

In particular, $D_1 \geqq 0$ on $\mathfrak{K}$.

Since $N = UP = PU$, where U and P are defined in (1.2) and (1.5), it is seen that (2.2) becomes $QP^2 - UPQPU^* = D_1$. Since $QP = PQ$ (by (1.3)) this becomes

$$QP^2 - U(QP^2)U^* = D_1 \quad (D_1 = D \oplus 0 \geqq 0), \tag{2.4}$$

where QP^2 is self-adjoint.

If Z denotes any Borel set on $|z|=1$ of measure 0, it follows from Theorem 2.3.2 of [9], p. 22, that $E(Z)D_1=0$ and hence, by (2.3), $E(Z)DQ=0$. Hence, for $k=0, 1, 2, \ldots$, $0=N^kE(Z)DQ=E(Z)N^kDQ=E(Z)T^kDQ$, and so $E(Z)x=0$ for any x in the least subspace of $\mathfrak{H}$ which is invariant under T and contains the range of D. Since T is completely subnormal, such a subspace must coincide with $\mathfrak{H}$, a fact observed by CLANCEY [2]. Thus $0=E(Z)Q=QE(Z)$ and hence $R_{E(Z)}\subset\mathfrak{H}^\perp=\mathfrak{K}\ominus\mathfrak{H}$. But $R_{E(Z)}$, hence also $\mathfrak{K}_1=\mathfrak{K}\ominus R_{E(Z)}$, reduces N. Since $\mathfrak{H}\subset\mathfrak{K}_1\subset\mathfrak{K}$ and since N is the minimal normal extension of T, it follows that $\mathfrak{K}_1=\mathfrak{K}$. Thus $E(Z)=0$, that is, U is absolutely continuous.

It remains to be shown that (1.4) holds. Suppose the contrary, that is, meas sp $(U)<2\pi$. It follows from (2.4) and Theorem 2.3.1 of [9], p. 21, that $\mathfrak{H}_a(QP^2)$ (note that $QP^2=P^2Q$ is self-adjoint) contains the least space, M, invariant under QP^2 and which also reduces U and contains $R_{D_1}(=R_D)$. Since $\mathfrak{H}$ (hence $\mathfrak{H}^\perp$) is invariant under QP^2 and $QP^2|\mathfrak{H}^\perp=0$, it follows that $\mathfrak{H}_a(QP^2)\subset\mathfrak{H}$ and hence $M\subset\mathfrak{H}$. Since $QP^2=P^2Q$, it is clear that M is invariant under P^2 and hence also under P. Since M also reduces U it follows that M reduces N. Further, since T is completely subnormal, hence not normal, $R_D\neq 0$ and, in particular, $M\neq 0$. Consequently, M is a non-trivial reducing space of T on which T is normal, so that T is not completely subnormal, a contradiction. Hence, meas sp $(U)=2\pi$, and the proof of Theorem 1 is now complete.

3. Proof of Theorem 2. Let $\mathfrak{H}_1$ denote the least subspace of $\mathfrak{K}$ containing $\mathfrak{H}$ and invariant under both N and N^*N, and let T_1 denote the restriction of N to $\mathfrak{H}_1$. Then T_1 is subnormal on $\mathfrak{H}_1$ with minimal normal extension N on $\mathfrak{K}$. It will be shown that $\mathfrak{H}_1=\mathfrak{K}$ (so that $T_1=N$). To see this, suppose, if possible, that $\mathfrak{H}_1$ is properly contained in $\mathfrak{K}$. Then T_1 is not normal and hence has a representation $T_1=T_{11}\oplus T_{12}$ on $\mathfrak{H}_1=\mathfrak{H}_{11}\oplus\mathfrak{H}_{12}$, where $\mathfrak{H}_{11}\neq 0$, T_{11} is completely subnormal on $\mathfrak{H}_{11}$, and, if $\mathfrak{H}_{12}$ is not empty, T_{12} is normal on $\mathfrak{H}_{12}$. Then $N=N_1\oplus T_{12}$ on $\mathfrak{K}=(\mathfrak{K}\ominus\mathfrak{H}_{12})\oplus\mathfrak{H}_{12}$, where N_1 is the minimal normal extension on $\mathfrak{K}\ominus\mathfrak{H}_{12}$ of T_{11} on $\mathfrak{H}_{11}$. Further, $\mathfrak{H}_{11}$ is invariant under N_1 and $N_1^*N_1$.

Clearly, sp $(N_1)\subset$ sp (N) and hence, by (1.7),

$$\text{sp}\,(N_1)\cap W \text{ is empty.} \tag{3.1}$$

It is seen that the relation corresponding to (1.3) of Theorem 1 now holds with T, N, $\mathfrak{H}$ and $\mathfrak{K}$ replaced by T_{11}, N_1, $\mathfrak{H}_{11}$ and $\mathfrak{K}\ominus\mathfrak{H}_{12}$ respectively. Hence sp $(U_1)=\{z: |z|=1\}$, where U_1 corresponds to N_1 as U does to N, in contradiction with (3.1). Consequently, $\mathfrak{H}_1=\mathfrak{K}$ and Theorem 2 is proved.

4. An example. Let m denote the measure on the set

$$S=\{z:|z|=1\}\cup\{0\}, \tag{4.1}$$

which is normalized Lebesgue measure on $|z|=1$ and is 1 at $z=0$. Let N be the position operator $(Nf)(z)=zf(z)$ on the Hilbert space $\mathfrak{K}=L^2(m)$ and let T denote the restriction of N to the space $\mathfrak{H}=H^2(m)$, the subspace of $L^2(m)$ spanned by $\{z^n\}$, $n=0, 1, 2, \ldots$. (This example is given in HALMOS [6], p. 309; see also STAMPFLI [11], p. 379. For a discussion of position operators see [9], pp. 15 ff.) Then T is subnormal with the minimal normal extension N. An orthonormal basis for $\mathfrak{H}=H^2(m)$ is $\{e_n(z)\}$, where $e_0(z)=1/2^{\frac{1}{2}}$ and $e_n(z)=z^n$ for $n=1, 2, \ldots$. Also $Te_0=(1/2^{\frac{1}{2}})e_1$ and $Te_n=e_{n+1}$ for $n=1, 2, \ldots$, so that T is the unilateral weighted shift with weights $\{1/2^{\frac{1}{2}}, 1, 1, \ldots\}$. Further, sp$(T)$ is the closed unit disk while sp(N) is the set S of (3.1). In particular, 0 is in the point spectrum of N and hence N cannot be radially absolutely continuous.

It follows from Theorem 1 that (1.3) cannot hold. This fact is also easily verified directly (note that N^*N is the multiplication operator $|z|^2$). It is seen that the operator N can be written as the direct sum $N=0\oplus N_1$ on $\mathfrak{K}=\mathfrak{K}_0\oplus\mathfrak{K}_1$, where $\mathfrak{K}_0$ is the eigenspace of N for $z=0$. (A basis for $\mathfrak{K}_0$ is the function which equals 1 at $z=1$ and equals 0 on $|z|=1$.) Further, N_1 is unitary and is absolutely continuous on $\mathfrak{K}_1$. In the context of Theorem 1 this can be explained by noting that N_1 is the minimal (unitary) extension of T_1: $(T_1f)(z)=zf(z)$ on $\mathfrak{H}_1=H^2(m_1)$ where m_1 is normalized Lebesgue measure on $|z|=1$.

5. Another example. Let D be a domain (non-empty connected open subset of the plane) and consider the Hilbert space $\mathfrak{H}=A^2(D)$ of functions analytic on D and square integrable with respect to ordinary Lebesgue planar measure; cf. [9], p. 15. Let T denote the position operator $(Tf)(z)=zf(z)$ for $f\in\mathfrak{H}=A^2(D)$ and let N denote its (minimal) normal extension $(Nf)(z)=zf(z)$ for $f\in\mathfrak{K}=L^2(D)$. Then

$$\text{(5.1)} \qquad \operatorname{sp}(T)=\operatorname{sp}(N)=\text{closure of } D,$$

and, in addition, N is radially absolutely continuous. In fact, N is even absolutely continuous in the (stronger) ordinary two-dimensional sense, that is, if N has the spectral resolution (1.1), then

$$\text{(5.2)} \qquad K(Z)=0 \quad \text{whenever } Z \text{ is a Borel set of planar measure } 0.$$

It is seen that condition (1.3) is not fulfilled, since if $f(z)$ is analytic on K, the function $|z|^2f(z)$ is not analytic unless $f(z)\equiv 0$. Nevertheless, Theorem 2 can be applied to yield

Theorem 3. *Let D be a domain for which there exists an open wedge of* (1.6) *satisfying*

$$\text{(5.3)} \qquad D\cap W \text{ is empty.}$$

Let $\mathfrak{H}_0(D)$ denote the Hilbert space obtained by taking the closure of the linear manifold of functions $\left\{\sum_{k=0}^{N} |z|^{2k} f_k(z)\right\}$, $N = 0, 1, \ldots$, *where the* $f_k(z)$ *are in* $A^2(D)$. *Then* $\mathfrak{H}_0 = L^2(D)$.

In fact, $\mathfrak{H}_0(D)$ is clearly the least subspace of $L^2(D)$ containing $\mathfrak{H}=A^2(D)$ and invariant under $N=z$ and $N^*N=|z|^2$. (Note also that the space $\mathfrak{H}_0(D)$ remains unchanged if, in its definition, $|z|^2$ is replaced by $|z|$.)

If (5.3) is not satisfied, the assertion of Theorem 3 can be false. For instance, if $D=\{z: |z|<1\}$, then $\mathfrak{H}_0(D)$ is a proper subspace of $L^2(D)$. In fact, one can here restrict the $f_k(z)$ to be polynomials in z. It is then easily verified that the space spanned by $\{\bar{z}^n\}$, $n=1, 2, \ldots$, is contained in the orthogonal complement $\mathfrak{H}_0^{\perp}(D)= =L^2(D)\ominus\mathfrak{H}_0(D)$.

6. Remarks. As noted above, a normal operator N of (1.1) is absolutely continuous (in the two-dimensional sense) if (5.2) holds. The question arises as to what conditions might assure this type of absolute continuity of the minimal normal extension of a subnormal operator. One answer can be given as follows. As before, suppose that T is completely subnormal on $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K}$, and suppose that (1.3) holds. This guarantees, in particular, that N is radially absolutely continuous. It turns out that if, for instance, in addition to (1.3),

$$NQ = NA - AN \tag{6.1}$$

holds for some bounded operator A on $\mathfrak{K}$, then N is necessarily absolutely continuous.

To see this, let $[A, B]=AB-BA$ for any pair of bounded operators A and B on a Hilbert space (here, $\mathfrak{K}$), so that (2.2) becomes $[QN^*, N]=D_1$. By (6.1), $QN^*= =[A^*, N^*]$ and so

$$[[A^*, N^*], N] = D_1 \geqq 0. \tag{6.2}$$

An argument similar to that of [9], pp. 24—25 (see also [8]) then shows that $K(Z)D_1=0$ where Z is a Borel set of planar measure 0 and D_1 is the non-negative operator of (2.3). An argument like that of section 2 above then implies (5.2).

Similar reasoning shows that, instead of (5.1), one could suppose

$$\text{either } NQ+B = NA-AN \quad \text{or} \quad QN^*+B_1 = NA_1-A_1N, \tag{6.3}$$

where A or A_1 denote arbitrary bounded operators and B or B_1 denote operators commuting with N (hence, by Fuglede's theorem, also with N^*).

That a second order commutator equation such as occurs in (6.2) with D_1 non-negative should arise in connection with two-dimensional absolute continuity of a normal operator is not unnatural. The situation is analogous to that of an ordinary commutator and one-dimensional absolute continuity of a self-adjoint or unitary operator; cf. [9], Chapter II, also KATO [7], p. 543. Concerning the solution of commutator equations $[A, X]=C$, where A is self-adjoint, see also ROSENBLUM [10], and, where A is normal or an infinitesimal generator of a certain strongly continuous semigroup, see FREEMAN [3], [4].

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