Bands of monoids

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To my colleagues in the city of Szeged where this paper has been written

Let S be a semigroup which is the union of a family $(S_i)_{i\in I}$ of subsemigroups which are classes of a congruence relation on S. Then I may be endowed with a binary operation $ij=k \leftrightarrow S_i S_j \subset S_k$ for all $i, j, k \in I$. Under this operation I is a *band* (i.e. an idempotent semigroup) and S is called an *I-band* (or merely a band) of subsemigroups $(S_i)_{i\in I}$.

In this paper we present a new method of constructing bands of semigroups. This method permits to build up all bands of unipotent monoids (a *monoid* is a semigroup with identity, a monoid is called *unipotent* if it contains the only idempotent — its identity). In particular, we obtain a simple construction for orthodox bands of arbitrary monoids. Our method is a generalization of Clifford's sums of direct systems of groups [1] (called also rigid or strong semilattices of groups).

In our paper [2] we introduced a class of semigroups with the weak involutory property (WIP-semigroups). A semigroup S is a WIP-semigroup if for any $s, t \in S$ and any $\bar{s}, \bar{t} \in S$ such that $s\bar{s}s=s$, $\bar{s}s\bar{s}=\bar{s}$, $t\bar{t}t=t$, $\bar{t}t\bar{t}=\bar{t}$ (i.e. $\bar{s}$ and $\bar{t}$ are inverses for s and t respectively), $\bar{t}\bar{s}$ is an inverse for st. Among other properties it was proved that S is a WIP-semigroup if and only if the idempotents of S form a (possibly empty) subsemigroup [2]. Regular WIP-semigroups were considered also in [3] where they were called orthodox semigroups. So we call the WIP-semigroups *orthodox* (notice that an orthodox semigroup in our sense need not be regular).

Let $(S_i)_{i\in I}$ be a family of semigroups with pairwise disjoint sets of elements. Suppose $\leqq$ is a quasiorder (i.e. reflexive and transitive) binary relation on I. A family $\Phi=(\varphi_{ij})_{i\leqq j}$; $i, j \in I$ is called a *direct system of homomorphisms over* $\leqq$ if for every $i, j \in I$ such that $i \leqq j$ φ_{ij} is a homomorphism of S_j into S_i and the following two properties holds:

1) for every $i \in I$ φ_{ii} is the identical automorphism of S_i;

2) for every $i, j, k \in I$ if $i \leqq j \leqq k$ then $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$.

If S_i are monoids and e_i denotes the identity of S_i then we demand that $\varphi_{ij}(e_j)=e_i$, i.e. identities are preserved under homomorphisms of monoids.

Let I be endowed with an associative and idempotent binary operation $\cdot$, i.e. let $(I, \cdot)$ be a band. Define the following binary relations $\leqq_1$ and $\leqq_2$ on I: $i \leqq_1 j \leftrightarrow ji=i$; $i \leqq_2 j \leftrightarrow ij=i$. Clearly, both $\leqq_1$ and $\leqq_2$ are quasiorder relations on I. Suppose $\Phi=(\varphi_{ij})$ and $\Psi=(\psi_{ij})$ are direct systems of homomorphisms over $\leqq_1$ and $\leqq_2$ respectively. Φ and Ψ are called *commuting* if for all $i, j, k \in I$ such that $j \leqq_1 i$, $k \leqq_2 i$ the following diagram is commutative:

$$\begin{array}{ccc} S_i & \rightarrow & S_j \\ \downarrow & & \downarrow \\ S_k & \rightarrow & S_{kj} \end{array}$$

where the horizontal arrows represent homomorphisms from Φ and vertical arrows represent homomorphisms from Ψ (i.e. $\psi_{kj,j} \circ \varphi_{ji}=\varphi_{kj,k} \circ \psi_{ki}$). Clearly, $kj \leqq_1 k$ and $kj \leqq_2 j$ so that all homomorphisms mentioned do exist.

If $a_i \in S_i$ then ϱ_{a_i} and λ_{a_i} denote the right and left translations of S_i corresponding to a_i, i.e. $\varrho_{a_i}(s)=sa_i$ and $\lambda_{a_i}(s)=a_i s$ for all $s \in S_i$.

Suppose there are given two direct systems of homomorphisms Φ and Ψ over $\leqq_1$ and $\leqq_2$ respectively and an $(I \times I)$-matrix $A=(a_{ij})$ over $S=\bigcup_{i\in I} S_i$ such that $a_{ij} \in S_{ij}$ for all $i, j \in I$. We call the triple (Φ, Ψ, A) *balanced* if $a_{ii}=e_i$ for any $i \in I$ and

$$\varrho_{a_{ij,k}} \circ \varphi_{ijk,ij} \circ \lambda_{a_{ij}} \circ \psi_{ij,j} = \lambda_{a_{i,jk}} \circ \psi_{ijk,jk} \circ \varrho_{a_{jk}} \circ \varphi_{jk,j}$$

for all $i, j, k \in I$.

If $a_{ij}=e_{ij}$ for all $i, j \in I$ then the triple (Φ, Ψ, A) is balanced precisely if the direct systems Φ and Ψ commute.

A band S of monoids $(S_i)_{i\in I}$ is called *proper* if the identities of the monoids form a subsemigroup of S.

Theorem 1. *Let $(S_i)_{i\in I}$ be a family of pairwise disjoint unipotent monoids, $(I, \cdot)$ be a band, $\Phi=(\varphi_{ij})$ and $\Psi=(\psi_{ij})$ be direct systems of homomorphisms over $\leqq_1$ and $\leqq_2$ respectively, A be an $(I \times I)$-matrix over $S=\bigcup_{i\in I} S_i$ and the triple (Φ, Ψ, A) be balanced.*

Define a binary multiplication $\square$ on S as follows: if $s_i \in S_i$ and $s_j \in S_j$ then $s_i \square s_j=\varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j)$ where the right-hand side product is taken in the monoid S_{ij}. Then $(S, \square)$ is an I-band of monoids $(S_i)_{i\in I}$ and every I-band of monoids $(S_i)_{i\in I}$ can be constructed in this way. Moreover, the triple (Φ, Ψ, A) is defined uniquely for any I-band of $(S_i)_{i\in I}$.

Theorem 2. *Let $(S_i)_{i\in I}$ be a family of pairwise disjoint semigroups, $(I, \cdot)$ be a band, and Φ and Ψ be commuting direct systems of homomorphisms over $\leqq_1$ and*

$\leqq_2$, respectively. Define a binary multiplication $\square$ on $S=\bigcup_{i\in I}S_i$ as follows: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\square s_j=\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j)$ where the right-hand side product is taken in the semigroup S_{ij}. Then $(S, \square)$ is an I-band of semigroups $(S_i)_{i\in I}$. Moreover, if S_i are monoids then $(S, \square)$ is a proper I-band of the monoids $(S_i)_{i\in I}$ and every proper I-band of the monoids $(S_i)_{i\in I}$ can be constructed in the above fashion, the direct systems Φ and Ψ being determined in the unique way. $(S, \square)$ is orthodox if and only if all the monoids S_i are orthodox.

Some corollaries will follow after the proofs.

Proof of Theorem 1. Suppose $s_i\in S_i$, $s_j\in S_j$ and $s_k\in S_k$. Then

$$(s_i\square s_j)\square s_k = (\varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j))\square s_k = \varphi_{ijk,ij}(\varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) =$$

$$= [\varphi_{ijk,ij}\circ\varphi_{ij,i}(s_i)][\varrho_{a_{ij},k}\circ\varphi_{ijk,ij}\circ\lambda_{a_{ij}}\circ\psi_{ij,j}(s_j)]\psi_{ijk,k}(s_k) =$$

$$= \varphi_{ijk,i}(s_i)[\lambda_{a_{i,jk}}\circ\psi_{ijk,jk}\circ\varrho_{a_{jk}}\circ\varphi_{jk,j}(s_j)][\psi_{ijk,jk}\circ\psi_{jk,k}(s_k)] =$$

$$=\varphi_{ijk,i}(s_i)a_{i,jk}[\psi_{ijk,jk}(\varphi_{jk,j}(s_j)a_{jk}\psi_{jk,k}(s_k))]=\varphi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(s_j\square s_k)=s_i\square(s_j\square s_k),$$

i.e. $(S, \square)$ is a semigroup. If $i=j$ then $s_i\square s_j=\varphi_{ii}(s_i)a_{ii}\psi_{ii}(s_j)=s_ie_is_j=s_is_j$. Thus, $(S, \square)$ is an I-band of the family $(S_i)_{i\in I}$ of monoids.

Now $e_i\square e_j=\varphi_{ij,i}(e_i)a_{ij}\psi_{ij,j}(e_j)=e_{ij}a_{ij}e_{ij}=a_{ij}$ so that the matrix A is determined in the unique way — $A=(e_i\square e_j)$. Using this fact we obtain

$$a_{i,ij} = e_i\square e_{ij} = e_i\square(e_i\square e_{ij}) = e_i\square(e_{ij}\square(e_i\square e_{ij})) = (e_i\square e_{ij})^2,$$

i.e. $a_{i,ij}$ is an idempotent from S_{ij}. Since S_{ij} is unipotent, $a_{i,ij}=e_{ij}$. Thus,

$$s_i\square e_{ij} = \varphi_{ij,i}(s_i)a_{i,ij}\psi_{ij,ij}(e_{ij}) = \varphi_{ij,i}(s_i)\cdot a_{i,ij}e_{ij} = \varphi_{ij,i}(s_i)$$

i.e. the direct system Φ of homomorphisms is determined in the unique way. Analogously we may prove that $\psi_{ij,j}(s_j)=e_{ij}\square s_j$ for any $s_j\in S_j$.

To prove the second part of Theorem 1 suppose $(S, \cdot)$ is a band of a family $(S_i)_{i\in I}$ of unipotent monoids. Let $a_{ij}=e_ie_j$ for any $i,j\in I$, $\varphi_{ij,i}(s_i)=s_ie_{ij}$ and $\psi_{ij,j}(s_j)=e_{ij}s_j$ for all $i,j\in I$ and $s_i\in S_i$, $s_j\in S_j$. Then $a_{ij}\in S_{ij}$ and if s_i, $t_i\in S_i$ then

$$\varphi_{ij,i}(s_it_i) = s_it_ie_{ij} = s_i(e_{ij}(t_ie_{ij})) = \varphi_{ij,i}(s_i)\varphi_{ij,i}(t_i),$$

i.e. $\varphi_{ij,i}$ is a homomorphism of S_i into S_{ij}. Since S_{ij} is unipotent, $\varphi_{ij,i}(e_i)=e_{ij}$. Clearly $\varphi_{ii}(s_i)=s_ie_i=s_i$. Now

$$\varphi_{ijk,ij}\circ\varphi_{ij,i}(s_i) = \varphi_{ijk,ij}(s_ie_{ij}) = (s_ie_{ij})e_{ijk} = s_i(e_{ij}e_{ijk}) = s_ie_{ijk} = \varphi_{ijk,i}(s_i)$$

so that $\Phi=(\varphi_{ij})$ forms a direct system of homomorphisms over $\leqq_1$. In the same way we may prove that $\Psi=(\psi_{ij})$ forms a direct system of homomorphisms over $\leqq_2$.

Now $a_{ii}=e_ie_i=e_i$ and

$$\varrho_{a_{ij},k}\circ\varphi_{ijk,ij}\circ\lambda_{a_{ij}}\circ\psi_{ij,j}(s_j) = \varrho_{a_{ij},k}\circ\varphi_{ijk,ij}\circ\varrho_{a_{ij}}(e_{ij}s_j) = \varrho_{a_{ij},k}\circ\varphi_{ijk,ij}(a_{ij}e_{ij}s_j) =$$
$$= \varrho_{a_{ij},k}(a_{ij}e_{ij}s_je_{ijk}) = a_{ij}e_{ij}s_je_{ijk}a_{ij,k} = a_{ij}s_je_{ijk}a_{ij,k} = a_{ij}s_ja_{ij,k} =$$
$$= e_ie_js_ja_{ij,k} = e_is_ja_{ij,k} = e_is_je_{ij}e_k = e_is_je_k = e_ie_{jk}s_je_k = a_{i,jk}s_je_k =$$
$$= a_{i,jk}s_je_je_k = a_{i,jk}s_ja_{jk} = a_{i,jk}e_{ijk}s_ja_{jk} = a_{i,jk}e_{ijk}s_je_{jk}a_{jk} =$$
$$= \lambda_{a_{i,jk}}(e_{ijk}s_je_{jk}a_{jk}) = \lambda_{a_{i,jk}}\circ\psi_{ijk,jk}(s_je_{jk}a_{jk}) =$$
$$= \lambda_{a_{i,jk}}\circ\psi_{ijk,jk}\circ\varrho_{a_{jk}}(s_je_{jk}) = \lambda_{a_{i,jk}}\circ\psi_{ijk,jk}\circ\varrho_{a_{jk}}\circ\varphi_{jk,j}(s_j),$$

i.e. the triple (Φ, Ψ, A) is balanced. Finally

$$s_i\Box s_j = \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j) = s_ie_{ij}a_{ij}e_{ij}s_j = s_ia_{ij}s_j = s_ie_ie_js_j = s_is_j.$$

This fact completes the proof of Theorem 1.

Proof of Theorem 2. Suppose $s_i\in S_i$, $s_j\in S_j$ and $s_k\in S_k$. Then

$$(s_i\Box s_j)\Box s_k = (\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j))\Box s_k = \varphi_{ijk,ij}(\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j))\psi_{ijk,k}(s_k) =$$
$$= [\varphi_{ijk,ij}\circ\varphi_{ij,i}(s_i)][\varphi_{ijk,ij}\circ\psi_{ij,j}(s_j)]\psi_{ijk,k}(s_k) =$$
$$= \varphi_{ijk,i}(s_i)[\psi_{ijk,jk}\circ\varphi_{jk,j}(s_j)][\psi_{ijk,jk}\circ\psi_{jk,k}(s_k)] = \varphi_{ijk,i}(s_i)\psi_{ijk,jk}(s_j\Box s_k) = s_i\Box(s_j\Box s_k),$$

i.e. $(S, \Box)$ is a semigroup.

If $i=j$ then $s_i\Box s_j=\varphi_{ii}(s_i)\psi_{ii}(s_j)=s_is_j$. Thus, $(S, \Box)$ is an I-band of the family $(S_i)_{i\in I}$ of semigroups. Unicity of Φ and Ψ in case S are monoids for all $i\in I$ is proved in the same way as in the proof of Theorem 1.

If $(S, \cdot)$ is a proper I-band of monoids S_i then exactly in the same way as in the proof of Theorem 1 we may verify that $(S, \cdot)=(S, \Box)$ where Φ and Ψ are defined in the same way as in the proof of Theorem 1. Commutativity of Φ and Ψ follows readily.

If S_i are monoids then $e_i\Box e_j=\varphi_{ij,i}(e_i)\psi_{ij,j}(e_j)=e_{ij}e_{ij}=e_{ij}$. Therefore $(S, \Box)$ is a proper band of $(S_i)_{i\in I}$.

Clearly, if $(S, \Box)$ is orthodox then S_i are orthodox for all $i\in I$. Conversely, suppose S_i are orthodox and $s_i\in S_i$, $s_j\in S_j$ are idempotents of $(S, \Box)$. Then $s_i\Box s_j=$ $=\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j)$ and the right-hand side of the equality is a product of two idempotents of S_{ij} (since homomorphisms map idempotents onto idempotents). The orthodoxy of S_{ij} implies $s_i\Box s_j$ is an idempotent. Thus, $(S, \Box)$ is orthodox which completes the proof of Theorem 2.

Obviously, Theorem 2 in case of unipotent monoids is a particular case of Theorem 1.

Remark 1. Since every group is a unipotent and orthodox monoid, every band of groups may be constructed as in Theorem 1 and every orthodox band of groups may be constructed as in Theorem 2. Another construction for orthodox bands of groups has been given in [4]. A survey of constructions for orthodox unions of groups may be found in [5].

Remark 2. Suppose (Φ, Ψ, A) is a balanced triple and $k \leqq_1 j$, $i \leqq_2 j$. This being the case, $a_{ij}=e_i$ (which fact has been proved above) and analogously $a_{jk}=e_k$. Thus, the condition of balancedness may be written for these particular i, j, k as follows:

$$(1) \qquad \varrho_{a_{ik}} \circ \varphi_{ik,i} \circ \psi_{ij} = \lambda_{a_{ik}} \circ \psi_{ik,k} \circ \varphi_{kj}.$$

If $i=k$ then we obtain $\varrho_{a_{ii}} \circ \varphi_{ii} \circ \psi_{ij} = \lambda_{a_{ii}} \circ \psi_{ii} \circ \varphi_{ij}$ or, equivalently, $\psi_{ij}=\varphi_{ij}$. Thus, if $i \leqq_1 j$ and $i \leqq_2 j$ (i.e. if $i=ij=ji$) then $\psi_{ij}=\varphi_{ij}$. In particular, if $(I, \cdot)$ is a semilattice then $\leqq_1$ coincides with $\leqq_2$ and Φ coincides with Ψ; in this case the construction of Theorem 2 turns out to be the well-known [1] construction for sums of direct systems of semigroups. Clearly, if $\Phi=\Psi$ then Φ and Ψ commute. Thus, *every proper semilattice of monoids is a sum of their direct system.*

Remark 3. Let the band $(I, \cdot)$ satisfy the pseudoidentity $xyx=xy \vee xyx=yx$ where $\vee$ is the disjunction sign. Let $x \leqq y$ mean that $x \leqq_1 y$ or $x \leqq_2 y$. Then $\leqq$ is a quasiorder relation on I. In effect, $\leqq$ is obviously reflexive. To show transitivity of $\leqq$, suppose $i \leqq j$ and $j \leqq k$ for some $i, j, k \in I$. Suppose $i \leqq_1 j$. If $j \leqq_1 k$ then $i \leqq_1 k$ and $i \leqq_1 k$, so let $j \leqq_2 k$. Then $ji=i$ and $jk=j$. Then $iki=ki$ or $iki=ik$. If $iki=ki$ then $i=ji=(jk)i=j(ki)=j(iki)=(ji)ki=iki=ki$ and $i \leqq_1 k$, whence $i \leqq k$. If $iki=ik$ then $i=ji=(jk)i=j(ki)=j(kiki)=(jk)iki=(jk)ik=jik=ik$ and $i \leqq_2 k$, whence $i \leqq k$. Analogously, $i \leqq_2 j$ implies $i \leqq k$. Therefore, $\leqq$ is a quasiorder relation.

Conversely, suppose $\leqq$ is a quasiorder relation. Then the band $(I, \cdot)$ satisfies the above pseudoidentity. In effect, for every two elements $x, y \in I$ the relations $xyx \leqq_1 xy$ and $xy \leqq_2 y$ hold in every band. Therefore, $xyx \leqq xy \leqq y$ and, since $\leqq$ is transitive, $xyx \leqq y$, i.e. $xyx \leqq_1 y$ or $xyx \leqq_2 y$. The latter means that $xyx=y(xyx)=$ $=(yx)^2=yx$ or $xyx=(xyx)y=(xy)^2=xy$, i.e. $xyx=xy \vee xyx=yx$.

Two quasiorder relations on a same set are called *compatible* if their set-theoretical union is a quasiorder relation. We have proved the following

Lemma 1. *A band satisfies the pseudoidentity $xyx=xy \vee xyx=yx$ if and only if its quasiorder relations $\leqq_1$ and $\leqq_2$ are compatible.*

Now if $i \leqq j$ then either $i \leqq_1 j$ or $i \leqq_2 j$ or both. Suppose two direct systems of homomorphisms Φ and Ψ over $\leqq_1$ and $\leqq_2$ respectively are given. Then φ_{ij} or ψ_{ij} is defined. If both homomorphisms are defined then $i \leqq_1 j$ and $i \leqq_2 j$ which implies, as we have seen in Remark 2, $\varphi_{ij}=\psi_{ij}$. Therefore, one may consider the system $X=(\chi_{ij})_{i \leqq j;\, i, j \in I}$ of homomorphisms: χ_{ij} coincides with that of homomorphisms φ_{ij}, ψ_{ij} which is defined.

Let the above pseudoidentity be satisfied and $(S, \cdot)$ be an I-band of the family $(S_i)_{i\in I}$ of monoids. If $i\leqq_1 j$, i.e. if $ji=i$, then, as we have seen above, $e_je_i=e_j(e_je_i)=$ $=e_j(e_i(e_je_i))=(e_je_i)^2$. Suppose now all S_i are unipotent. Then $e_je_i=e_i$. Analogously $i\leqq_2 j$ implies $e_ie_j=e_i$. Now let i and j be arbitrary elements of I. Then either $iji=ij$ or $iji=ji$. In the first case $e_ie_j\in S_{ij}$, therefore $e_{ij}e_ie_j=e_ie_j$. Now

$$e_{ij}e_i\in S_{ij}S_i\subset S_{iji}=S_{ij},$$

therefore

$$e_{ij}e_i=(e_{ij}e_i)e_{ij}=e_{ij}(e_ie_{ij})=e_{ij}e_{ij}=e_{ij},$$

since $ij\leqq_1 i$. Hence

$$e_ie_j=e_{ij}e_ie_j=e_{ij}e_j=e_{ij},$$

since $ij\leqq_2 j$.

Suppose now

$$iji=ji.$$

Then

$$ij=(ij)^2=(iji)j=(ji)j \quad \text{and} \quad e_je_{ij}\in S_{jij}=S_{ij}.$$

It follows that

$$e_je_{ij}=e_{ij}(e_je_{ij})=(e_{ij}e_j)e_{ij}=e_{ij}e_{ij}=e_{ij}$$

and

$$e_ie_j=(e_ie_j)e_{ij}=e_i(e_je_{ij})=e_ie_{ij}=e_{ij}.$$

Thus,

$$a_{ij}=e_ie_j=e_{ij}$$

for any $i, j\in I$, i.e. $(S, \cdot)$ is a proper band of monoids. Then the direct systems Φ and Ψ commute.

Now let $i\leqq j\leqq k$. If $i\leqq_1 j\leqq_1 k$ then $\chi_{ik}=\varphi_{ik}=\varphi_{ij}\circ\varphi_{jk}=\chi_{ij}\circ\chi_{jk}$. Analogously, $\chi_{ik}=\chi_{ij}\circ\chi_{jk}$ in case when $i\leqq_2 j\leqq_2 k$. Now let $i\leqq_1 j\leqq_2 k$. Then, as we have seen above, $i\leqq k$, i.e. $i\leqq_1 k$ or $i\leqq_2 k$. If $i\leqq_1 k$ then $\chi_{ik}=\varphi_{ik}$ and for every $s_k\in S_k$

$$\chi_{ik}(s_k)=\varphi_{ik}(s_k)=s_ke_i=e_i(s_ke_i)=(e_je_i)(s_ke_i)=e_j(e_i(s_ke_i))=e_j(s_ke_i)=$$
$$=(e_js_k)e_i=\varphi_{ij}\circ\psi_{jk}(s_k)=\chi_{ij}\circ\chi_{jk}(s_k),$$

i.e. $\chi_{ik}=\chi_{ij}\circ\chi_{jk}$. The same equality can be proved analogously if $i\leqq_2 j\leqq_1 k$. Since χ_{ii} is obviously the identical automorphism of S_i and X preserves identities of our monoids, X is a direct system of homomorphisms over $\leqq$.

The above argument together with Theorems 1 and 2 yields the following

Proposition 1. *Suppose $(I, \cdot)$ is a band satisfying the pseudoidentity $xyx=$ $=xy\vee xyx=yx$. Define $i\leqq j$ if and only if $i=iji$. Then $\leqq$ is a quasiorder relation, the set-theoretical union of the quasiorder relations $\leqq_1$ and $\leqq_2$ (i.e. $i\leqq j$ if and only if $i=ij$ or $i=ji$). Suppose $(S_i)_{i\in I}$ is a family of pairwise disjoint monoids and X is a direct system of homomorphisms over $\leqq$. Define a binary multiplication $\square$ on S as follows: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\square s_j=\chi_{ij,i}(s_i)\chi_{ij,j}(s_j)$ where the right-hand side product is taken inside the monoid S_{ij}. Then $(S, \square)$ is a proper I-band of the family*

$(S_i)_{i\in I}$ *of monoids and conversely, every proper I-band of these monoids can be constructed in the above way, the direct system X being determined in the unique fashion for each proper I-band of* $(S_i)_{i\in I}$. *Moreover, every I-band of unipotent monoids is necessarily proper (and hence orthodox) and so it can be constructed in the above way.*

In particular, Proposition 1 holds if $(I, \cdot)$ satisfies one of the following identities: $xyx=xy$, $xyx=yx$, $xyz=yxz$, $xyz=xzy$, $xy=x$, $xy=y$, $xy=yx$. In the latter case, i.e. for semilattices of unipotent monoids, this has been proved in [8].

It can be easily verified that $(I, \cdot)$ *satisfies the identity* $xyx=xy$ *[the identity* $xyx=yx$*] if and only if the quasiorder relation* $\leqq_1$ *[the quasiorder relation* $\leqq_2$*] is included into* $\leqq_2$ *[into* $\leqq_1$*].* Every band is a semilattice of rectangular bands. Right zero and left zero bands are called singular. It can be trivially verified that *a band satisfies the pseudoidentity* $xyx=xy \vee xyx=yx$ *if and only if it is a semilattice of singular bands.*

Remark 4. Suppose $(I, \cdot)$ is a rectangular band and $i,j\in I$. Then $i\leqq_1 ij$ and $ij\leqq_1 i$, whence $\varphi_{i,ij}\circ\varphi_{ij,i}=\varphi_{ii}$ and $\varphi_{ij,i}\circ\varphi_{i,ij}=\varphi_{ij,ij}$. Therefore, $\varphi_{i,ij}$ is an isomorphism. In the same way we may prove that $\psi_{j,ij}$ is an isomorphism. It follows that S_i and S_j are isomorphic. Thus, all the monoids S_i are pairwise isomorphic. This fact permits us to give an alternative construction for rectangular bands of unipotent monoids.

Fix some element $o\in I$ and for every $i\in I$ fix an isomorphism α_i of S_i onto S_o, say, $\alpha_i=\psi_{o,io}\circ\varphi_{io,i}$. If $s_i\in S_i$ let $f(s_i)=(\alpha_i(s_i), i)$. Then f is a bijective mapping of $S=\bigcup_{i\in I}S_i$ onto the Cartesian product of the sets S_o and I. It remains to define a suitable multiplication in $S_o\times I$ in order f to be an isomorphism. It is clear that

$$\alpha_i(s_i) = \psi_{o,io}\circ\varphi_{io,i}(s_i) = \psi_{o,io}(s_ie_{io}) = e_o(s_ie_{io})$$

so that $f(s_i)=(e_os_ie_{io}, i)$. Now suppose $(s,i)\in S_o\times I$. Then $f^{-1}((s,i))=e_{io}se_i$. In effect,

$$e_{io}se_i\in S_{io}S_oS_i\subset S_{iooi} = S_i \quad\text{and}\quad f(e_{io}se_i) = (e_oe_{io}se_ie_{io}, i) = (s,i)$$

since

$$e_oe_{io} = e_{o(io)}e_{io} = e_{o(io)} = e_o \quad\text{and}\quad se_ie_{io} = se_{io} = (se_o)e_{io} = s(e_oe_{io}) = se_o = s$$

so that

$$e_oe_{io}se_ie_{io} = e_os = s.$$

Thus, we should define such a multiplication $\square$ on $S_o\times I$ that for any $s,t\in S_o$ and any $i,j\in I$

$$(s,i)\square(t,j) = f((e_{io}se_i)\cdot(e_{jo}te_j)) = (e_o(e_{io}se_i)(e_{jo}te_j)e_{(ij)o}, ij).$$

Now

$$e_oe_{io}s = e_os = s \quad\text{and}\quad (ij)o = io$$

so that

$$e_o(e_{io}se_i)(e_{jo}te_j)e_{(ij)o} = se_ie_{jo}te_je_{io} = [s(e_oe_ie_{jo})][t(e_oe_je_{io})] = (sb_{ij})(tb_{ji})$$

where

$$b_{ij} = e_oe_ie_{jo} \in S_{oijo} = S_o.$$

Now

$$b_{ij}b_{ji,k} = (e_oe_ie_{jo})(e_oe_{ji}e_{ko}) = e_oe_i(e_{jo}e_o)e_{ji}e_{ko} = e_oe_ie_{jo}e_{ji}e_{ko} =$$

$$= e_oe_ie_{jo}(e_je_{ji})e_{ko} = e_oe_i(e_{jo}e_j)e_{ji}e_{ko} = e_oe_ie_je_{ji}e_{ko} = e_oe_i(e_je_{ji})e_{ko} =$$

$$= e_oe_ie_{ji}e_{ko} = e_o(e_ie_{ji})e_{ko} = e_oe_ie_{ko} = b_{ik},$$

since

$$e_{jo}e_j = e_j \quad \text{and} \quad e_ie_{ji} = e_i$$

which may be proved in the same way as the above equality $e_{oi}e_o = e_o$.

Conversely, suppose a unipotent monoid S_o and a rectangular band I are given and $b_{ij}b_{ji,k} = b_{ik}$ for every $i, j, k \in I$. Then $b_{ii}b_{ii} = b_{ii}b_{ii,i} = b_{ii}$ which implies that $b_{ii} = e_o$ for every $i \in I$. Now

$$b_{ij,i} = b_{ij,i}e_o = b_{ij,i}b_{ij,ij} = b_{ij,i}b_{i(ij),ij} = b_{ij,ij} = e_o,$$

whence

$$b_{i,jk}b_{jki,j} = b_{ij} \quad \text{and} \quad b_{i,jk}b_{jki,j} = b_{i,jk}b_{ji,j} = b_{i,jk}e_o = b_{i,jk},$$

i.e. $b_{i,jk} = b_{ij}$. On the Cartesian product $S_o \times I$ define the following multiplication $\square$: $(s, i) \square (t, j) = (sb_{ij}tb_{ji}, ij)$. Then $(s, i) \square (t, i) = (sb_{ii}tb_{ii}, ii) = (se_ote_o, i) = (st, i)$, i.e. $S_i = S_o \times \{i\}$ is isomorphic to S_o. Now

$$[(s, i) \square (t, j)] \square (u, k) = (sb_{ij}tb_{ji}, ij) \square (u, k) = (sb_{ij}tb_{ji}b_{ij,k}ub_{k,ij}, ijk) =$$

$$= (sb_{ij}tb_{jk}ub_{ki}, ijk) = (sb_{i,jk}tb_{jk}ub_{kj}b_{jk,i}, ijk) =$$

$$= (s, i) \square (tb_{jk}ub_{kj}, jk) = (s, i) \square [(t, j) \square (u, k)].$$

Thus, $(S_o \times I, \square)$ is an I-band of monoids isomorphic to S_o, namely, of monoids S_i.

We have proved the following

Proposition 2. *Let S be a unipotent monoid, I be a rectangular band, $B = (b_{ij})$ be an $(I \times I)$-matrix over S such that $b_{ij}b_{ji,k} = b_{ik}$ for all $i, j, k \in I$. Define the following multiplication $\square$ on the set $S \times I$: $(s, i) \square (t, j) = (sb_{ij}tb_{ji}, ij)$. Then $(S \times I, \square)$ is an I-band of monoids isomorphic to S and every I-band of monoids isomorphic to S can be constructed in the above way. In particular, there exists an I-band of a family $(S_i)_{i \in I}$ of unipotent monoids if and only if all the monoids are pairwise isomorphic.*

Another description of rectangular bands of unipotent monoids has been given in [9, Corollary 3.10].

In case of proper bands we have the following

Proposition 3. *Let $(S_i)_{i\in I}$ be a family of monoids and I be a rectangular band. There exists a proper I-band of $(S_i)_{i\in I}$ if and only if all the monoids are pairwise isomorphic, and every such band is isomorphic to a direct product of S_i for some fixed $i\in I$ and I. Conversely, every direct product of S_i and I is isomorphic to a proper I-band of $(S_i)_{i\in I}$.*

In effect, from Theorem 2 it follows that Φ and Ψ commute which implies easily our Proposition.

Another proof of Proposition 3 has been given in [6].

Since every band of semigroups is a semilattice of rectangular bands of semigroups [7], Proposition 2 gives some additional insight into the structure of bands of unipotent monoids and Proposition 3 — into the structure of proper bands of monoids.

In particular, if S is a *combinatorial* monoid (i.e. S has no invertible elements except 1 where 1 is the identity of S) then every I-band of monoids isomorphic to S is isomorphic to a direct product of S and I. This follows from the fact that b_{ij} is an invertible element of S for every $i;j\in I$. Moreover, $b_{ij}^{-1}=b_{ji,i}$. In effect, $b_{ij}b_{ji,i}=b_{ii}=1$ and $b_{ji,i}b_{ij}=b_{ji,i}b_{i(ji),j}=b_{ji,j}=b_{ji,ji}=1$.

It is a well-known fact that rectangular bands of groups are precisely the completely simple semigroups. Thus Proposition 2 gives, in particular, a new representation theorem for completely simple semigroups.

Remark 5. Suppose I is a band and $i\leqq j\leftrightarrow i=iji$. Then $\leqq$ is a quasiorder relation on I. Suppose $(S_i)_{i\in I}$ is a family of monoids and X is a direct system of homomorphisms over $\leqq$. Since $ij\leqq i$ and $ij\leqq j$ for every $i,j\in I$ we may define the following operation $\square$ on $S=\bigcup_{i\in I}S_i$: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\square s_j=\chi_{ij,i}(s_i)\chi_{ij,j}(s_j)$. Then $(S,\square)$ is an I-band of our monoids. A particular case of this construction was used in Proposition 1. Clearly, $(S,\square)$ is a proper I-band. Suppose $i\leqq j$. Then

$$e_i\square s_j\square e_i=\chi_{iji,i}(e_i)\chi_{iji,j}(s_j)\chi_{iji,i}(e_i)=\chi_{ii}(e_i)\chi_{ij}(s_j)\chi_{ii}(e_i)=e_i\chi_{ij}(s_j)e_i=\chi_{ij}(s_j)$$

for every $s_j\in S_j$.

Proposition 4. *Let I be a band satisfying the identity $xyxzx=xyzx$, $(S_i)_{i\in I}$ be a family of pairwise disjoint monoids and $i\leqq j\leftrightarrow i=iji$ for all $i,j\in I$. Define an operation $\square$ on the set $S=\bigcup_{i\in I}S_i$ as follows: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\square s_j=\chi_{ij,i}(s_i)\chi_{ij,j}(s_j)$. Then $(S,\square)$ is a proper I-band of monoids $(S_i)_{i\in I}$ and every proper I-band of these monoids can be constructed in the above way, the direct system X being determined uniquely for every proper I-band of $(S_i)_{i\in I}$.*

Proof. Suppose $(S, \cdot)$ is a proper I-band of $(S_i)_{i\in I}$. Then for every $i, j\in I$ such that $i\leqq j$ and every $s_j, t_j\in S_j$

$$(e_i s_j e_i)(e_i t_j e_i) = e_i s_j e_i t_j e_i = (e_i s_j) e_{ij} e_i e_{ji} (t_j e_i) = e_i s_j e_j e_{ij} e_i e_{ji} e_j t_j e_i =$$

$$= e_i s_j e_j e_{ij} e_{ji} e_j t_j e_i = e_i s_j e_{ij} e_{ji} t_j e_i = e_i s_j t_j e_i,$$

i.e. the mapping $\chi_{ij}\colon S_j \to S_i$ such that $\chi_{ij}(s_j) = e_i s_j e_i$ is a homomorphism. Clearly, $\chi_{ij}(e_j) = e_i e_j e_i = e_{iji} = e_i$ and χ_{ii} is the identical automorphism of S_i. Now let $i\leqq j\leqq k$ and $s_k\in S_k$. Then

$$\chi_{ij}\circ\chi_{jk}(s_k) = \chi_{ij}(e_j s_k e_j) = e_i e_j s_k e_j e_i = (e_i e_j) e_{kji} s_k e_j e_i = e_{ijkji} s_k e_{ji} = e_{iji} s_k e_{ji} =$$

$$e_i s_k = e_{ji} = e_i s_k e_{ik} e_{ji} = e_i s_k e_{ikji} = e_i s_k e_i = \chi_{ik}(s_k),$$

i.e. (χ_{ij}) form a direct system of homomorphism. We used the fact that $ikji = i$. In effect,

$$ikji = ikiji = ijikiji = ijkiji = ijkji = iji = i.$$

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