Fourier effective methods of summation

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1. Let the Fourier expansion of $f(x) \in L(-\pi, \pi)$ be

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=0}^{\infty} a_n(x).$$

We consider now summation of the series at a given point x. The summation behaviour of this series at a point x is reduced to properties of the cosine expansion

$$\varphi(t) \sim \sum_{n=0}^{\infty} a_n \cos nt$$

at $t=0$, where

$$\varphi(t) = \{f(x+t)+f(x-t)\}, \quad a_n = a_n(x).$$

By F_C we denote the class of all series $\sum a_n \cos nt$ for which $\varphi(t)$ is continuous at $t=0$, and by F_L the class of all series $\sum a_n \cos nt$ for which $t=0$ is a Lebesgue point of $\varphi(t)$, i.e.

$$\int_0^h |\varphi(t)-\varphi(0)|\,dt = o(h), \quad (0 < h \to 0).$$

W. B. JURKAT and A. PEYERIMHOFF [3] considered general summation methods $B=(b_{n\nu})$ in the series to sequence form satisfying

$$b_{n\nu} \to 1 \quad (n \to \infty,\ \nu \text{ fixed}), \quad b_{n\nu} \to 0 \quad (n \text{ fixed},\ \nu \to \infty).$$

$$\sum_{\nu=0}^{\infty} b_{n\nu} a_\nu = \sigma_n(\varphi) \quad (C, 1),$$

which means summable by the first Cesàro method. They called this the applicability condition. If for a method B satisfying the applicability condition

$$\sigma_n(\varphi) \to \varphi(0) \quad (n \to \infty)$$

for all φ corresponding to series F_C, respectively F_L, then we call the method B F_C-effective, respectively F_L-effective. Concerning F_C-effectiveness, they proved the following theorems.

Theorem A. *A method $B=(b_{n\nu})$ with the applicability property is F_C-effective if and only if*

$$\frac{1}{2}b_{n0}+\sum_{\nu=1}^{\infty}b_{n\nu}\cos\nu t \quad (n=0,1,\dots)$$

are the cosine expansions of functions (which are called kernels) $b_n(t)\in L(0,\pi)$ satisfying for every δ in $0<\delta<\pi$,

(i) $$\operatorname*{ess\,sup}_{\delta\le t\le\pi}|b_n(t)|\le M_\delta \quad (n=0,1,\dots),$$

(ii) $$\int_\delta^\pi b_n(t)\,dt\to 0,\quad \frac{2}{\pi}\int_0^\pi b_n(t)\,dt\to 1 \quad (n\to\infty),$$

(iii) $$\int_0^\pi |b_n(t)|\,dt\le M \quad (n=0,1,\dots).$$

Theorem B. *Let $\sum a_\nu$ be summable to the same s by all F_C-effective methods B. Then the series $\sum a_\nu\cos\nu t$ is the cosine expansion of a function $\varphi(t)\in L(0,\pi)$ which is continuous at $t=0$. In other words, the intersection of summability fields of all F_C-effective methods is F_C.*

In the present note, we will give the complete analogues of Theorems A and B for F_L-effectiveness.

2. Theorem 1. *A method $B=(b_{n\nu})$ with the applicability property is F_L-effective if and only if*

$$\frac{1}{2}b_{n0}+\sum_{\nu=1}^{\infty}b_{n\nu}\cos\nu t \quad (n=0,1,\dots)$$

are the cosine expansions of functions $b_n(t)\in L(0,\pi)$ satisfying for every δ $(0<\delta<\pi)$

(i) $$\operatorname*{ess\,sup}_{\delta\le t\le\pi}|b_n(t)|\le M_\delta \quad (n=0,1,\dots),$$

(ii) $$\int_\delta^\pi b_n(t)\,dt\to 0,\quad \frac{2}{\pi}\int_0^\pi b_n(t)\,dt\to 1 \quad (n\to\infty),$$

(iii) $$\int_0^\pi m_n(t)\,dt\le M,\quad \text{where}\quad m_n(t)=\operatorname*{ess\,sup}_{t\le u\le\pi}|b_n(u)|.$$

In other words, the kernel $b_n(t)$ has hump-backed majorants with uniformly bounded integrals.

Proof. Since F_L-effectiveness implies F_C-effectiveness, $(b_{n\nu})$ has to satisfy the condition of Theorem A. We write the kernel $b_n(t)$ as

$$b_n(t)\sim\frac{1}{2}b_{n0}+\sum_{\nu=1}^{\infty}b_{n\nu}\cos\nu t \quad (n=0,1,\dots).$$

If we can write

$$\sigma_n(\varphi) = \frac{2}{\pi}\int_0^\pi \varphi(t) b_n(t)\, dt \tag{1}$$

as a Lebesgue integral, a condition for F_L-effectiveness was given by D. FADDEEFF [2], see also S. G. KREIN—B. JA. LEVIN [4] and K. TANDORI [5]. The exposition is also given in ALEXITS' book [1].

For the representation (1), we proceed with Tandori's idea. Without loss of generality we can suppose $\varphi(0)=0$. Let us denote by L_0 the class of all functions $\varphi(t)\in L(0,\pi)$ satisfying $\varphi(0)=0$ and

$$\int_0^h |\varphi(t)|\, dt = o(h) \quad (0 < h \to 0).$$

Then Tandori proved that with the norm

$$\|\varphi\|_0 = \sup_{0<h\leqq\pi} \left\{\frac{1}{h}\int_0^h |\varphi(t)|\, dt\right\}$$

L_0 is the Banach space. For any fixed n, $\sigma_n(\varphi)$ is evidently a linear functional on L_0. We consider all functions which belong to $L(0,\pi)$ and vanish near the origin. This class is a subspace of L_0 and denoted by L_0^*. For any $\varphi\in L_0^*$

$$\sigma_n(\varphi) = \frac{2}{\pi}\int_0^\pi \varphi(t) b_n(t)\, dt.$$

In particular for any fixed n we take

$$\int_{\pi 2^{-m-1}}^{\pi 2^{-m}} |\varphi_m(t)|\, dt = 1,$$

$$\int_{\pi 2^{-m-1}}^{\pi 2^{-m}} \varphi_m(t) b_n(t)\, dt \geqq \operatorname*{ess\,sup}_{\pi 2^{-m-1}\leqq t\leqq \pi 2^{-m}} |b_n(t)| - \frac{\varepsilon}{\pi}$$

for any given $\varepsilon>0$ and set

$$\varphi^*(t) = \pi 2^{-m}\varphi_m(t) \quad \text{in} \quad (\pi 2^{-m-1}, \pi 2^{-m}) \quad (m = 0, 1, \dots),$$

$$\varphi^*(t) = 0 \quad \text{in} \quad (0, \pi 2^{-N}) \quad \text{for some} \quad N > m+1.$$

If we take $\pi 2^{-k-1}<h\leqq\pi 2^{-k}$ $(N>k+1)$, then

$$\frac{1}{h}\int_0^h |\varphi^*(t)|\, dt \leqq \frac{2^{k+1}}{\pi}\int_{\pi 2^{-N}}^{\pi 2^{-k}} |\varphi^*(t)|\, dt = \frac{2^{k+1}}{\pi}\sum_{m=k}^{N-1}\frac{\pi}{2^m}\int_{\pi 2^{-m-1}}^{\pi 2^{-m}} |\varphi_m(t)|\, dt =$$

$$= 2^{k+1}\sum_{m=k}^{N-1} 2^{-m} = 2^{k+1}(2^{-k+1} - 2^{-N+1}) = 4 - 2^{k-N+2} \leqq 4.$$

So $\|\varphi^*\|_0 \leqq 4$. On the other hand,

$$\sigma_n(\varphi^*) = \frac{2}{\pi}\int_0^\pi 2\varphi^*(t)b_n(t) \geqq \sum_{m=0}^{N} \frac{\frac{2}{\pi}}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1}\leqq t\leqq \pi 2^{-m}} |b_n(t)| - \varepsilon.$$

Hence we get

$$\sup_{\|\varphi^*\|_0\leqq 1} |\sigma_n(\varphi^*)| \geqq \frac{1}{2}\sum_{m=0}^{N}\frac{1}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1}\leqq t\leqq \pi 2^{-m}} |b_n(t)|$$

and we have

$$\sum_{m=0}^{\infty}\frac{1}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1}\leqq t\leqq \pi 2^{-m}} |b_n(t)| \sim \int_0^\pi \{\operatorname*{ess\,sup}_{t\leqq u\leqq \pi} |b_n(u)|\}\, dt$$

is finite for any fixed n. Thus the integral

$$\int_0^\pi |\varphi(t)b_n(t)|\,dt \leqq \sum_{m=0}^{\infty}\left\{2^m \int_{\pi 2^{-m-1}}^{\pi 2^{-m}} |\varphi(t)|\,dt\right\}\left\{\frac{1}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1}\leqq t\leqq \pi 2^{-m}} |b_n(t)|\right\}$$

exists in the Lebesgue sense for any $\varphi \in L_0$ as Tandori shows. We get the representation (1) by extension from L_0^* to L_0 and the conclusion is given by Faddeeff's theorem.

Theorem 2. *Let $\sum a_\nu$ be summable to the same s by all F_L-effective method B. Then $\sum a_\nu \cos \nu t$ is the cosine expansion of a function $\varphi \in L(0, \pi)$ which has $t=0$ as its Lebesgue point, i. e.*

$$\int_0^h |\varphi(t) - s|\,dt = o(h) \quad (0 < h \to 0).$$

In other words, the intersection of summability fields of all F_L-effective methods is F_L.

Proof. Fix the series

$$\frac{1}{2}a_0 + \sum_{\nu=0}^{\infty} a_\nu$$

and consider only kernels $b_n(t) \in C[0, \pi]$. By the same idea as in W. B. Jurkat and A. Peyerimhoff [3] we can prove that there exists a function $\varphi(t) \in L(0, \pi)$ such that

$$\frac{2}{\pi}\int_0^\pi \cos \nu t\, \varphi(t)\,dt = a_\nu \quad (\nu = 0, 1, 2, \ldots).$$

Hence for every bounded F_L-effective kernel $b_n(t)$ by Parseval's relation

$$\sigma_n(\varphi) = \frac{2}{\pi}\int_0^\pi b_n(t)\varphi(t)\,dt.$$

Next we have to show that

$$\text{(2)} \qquad \int_0^h |\varphi(t)-s|\,dt = o(h) \quad (0<h\to 0).$$

We can suppose $s=0$. If (2) fails, we may assume that some $\varepsilon>0$ and $h_k\to 0$ exist such that

$$\frac{1}{h_k}\int_0^{h_k} |\varphi(t)|\,dt > \varepsilon.$$

Set

$E_k=[0,h_k]$, $E_k^+=\{t|0\leqq t\leqq h_k, \varphi(t)\geqq 0\}$, and $E_k^-=\{t|0\leqq t\leqq h_k, \varphi(t)<0\}$, then

$$E_k^+\cup E_k^- = E_k, \quad E_k^+\cap E_k^- = \varphi, \quad \text{and} \quad |E_k^+|+|E_k^-| = h_k.$$

We select a subsequence $\{n_k\}$ such that

$$\alpha = \lim_{n_k\to\infty} |E_{n_k}^-|/|E_{n_k}^+|$$

exists $(0\leqq\alpha\leqq\infty)$. Let us set

$$\Psi_{E_{n_k}}(t) = \operatorname{sign}\varphi(t) \quad \text{for} \quad t\in E_{n_k}, \quad \text{and} \quad \Psi_{E_{n_k}} = 0 \quad \text{for} \quad t\notin E_{n_k},$$

and

$$d_{n_k}(t) = \frac{\pi}{2}\Psi_{E_{n_k}}(t)/(Ch_{n_k})$$

where C will be determined soon. Then,

$$\frac{2}{\pi}\int_0^\pi d_{n_k}(t)\,dt = \frac{2}{\pi}\int_0^{h_{n_k}} d_{n_k}(t)\,dt = \frac{1}{Ch_{n_k}}\int_0^{h_{n_k}} \Psi_{E_{n_k}}(t)\,dt$$

$$= \frac{|E_{n_k}^+|-|E_{n_k}^-|}{Ch_{n_k}} = \frac{1}{C}\frac{|E_{n_k}^+|-|E_{n_k}^-|}{|E_{n_k}^+|+|E_{n_k}^-|} \to \frac{1}{C}\frac{1-\alpha}{1+\alpha} \qquad (0\leqq\alpha\leqq\infty).$$

If $\alpha\neq 1$, we set $\dfrac{1}{C}=\dfrac{1+\alpha}{1-\alpha}$; then $d_{n_k}(t)$ satisfies (i) and (ii). The integral of the hump-backed majorant is

$$\frac{2}{\pi}\frac{\pi}{2}\int_0^\pi \frac{\chi_{E_{n_k}}(t)}{|C|h_{n_k}}\,dt = \frac{h_{n_k}}{|C|h_{n_k}} = \left|\frac{1+\alpha}{1-\alpha}\right| < \infty.$$

However,

$$\left|\frac{2}{\pi}\int_0^\pi \varphi(t)\,d_{n_k}(t)\,dt\right| =$$

$$= \frac{1}{|C|h_{n_k}}\int_0^{h_{n_k}} \varphi(t)\operatorname{sign}\varphi(t)\,dt = \frac{1}{|C|}\frac{1}{h_{n_k}}\int_0^{h_{n_k}} |\varphi(t)|\,dt > \frac{\varepsilon}{|C|}.$$

Now we approximate $d_{n_k}(t)$ by continuous $b_{n_k}(t)$ and obtain a contradiction.

When $\alpha=1$, the absolute values of both

$$\frac{1}{h_{n_k}}\int_0^{h_{n_k}} \varphi^+(t)\,dt \quad\text{and}\quad \frac{1}{h_{n_k}}\int_0^{h_{n_k}} \varphi^-(t)\,dt$$

are greater than $\varepsilon/2$, where $\varphi^+(t)$ and $\varphi^-(t)$ are the positive and negative parts of $\varphi(t)$ for large n_k. Since

$$|E_{n_k}^-|/|E_{n_k}^+| \to 1 \quad (n_k \to \infty,\ h_{n_k} \to 0),$$

the function

$$d_{n_k}(t) = \pi\chi_{E_{n_k}^+}(t)/h_{n_k}$$

satisfies conditions (i) and (ii) of Theorem 1. The integral of the hump-backed majorant is smaller than

$$\frac{2}{\pi}\pi\int_0^{h_{n_k}} \frac{1}{h_{n_k}}\,dt \leqq 2.$$

However, we also have

$$\frac{2}{\pi}\int_0^{\pi} \varphi(t)\,d_{n_k}(t)\,dt = \frac{2}{\pi}\pi\int_0^{\pi} \frac{\chi_{E_{n_k}^+}(t)}{h_{n_k}}\varphi(t)\,dt = \frac{2}{h_{n_k}}\int_0^{h_{n_k}} \varphi^+(t)\,dt > \varepsilon,$$

which is a contradiction. Hence we proved the theorem completely.

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