Uniformly bounded groups in finite W^*-algebras

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1. This note contains a proof of the fact that every uniformly bounded group of elements in a finite W^*-algebra [6] is similar to a unitary group. As an application, we also get a generalization of a result of ARVESON and JOSEPHSON [2, Theorem 2.4].

The problem of similarity between a uniformly bounded group and a unitary one has been conjectured in [3] for linear operators on Hilbert spaces, where it is solved for amenable groups (i.e. groups having an invariant mean). As it is known, there are finite W^*-algebras whose unitary groups do not have any invariant mean in the sense of [3]. We give in what follows an answer for any uniformly bounded group in a finite W^*-algebra.

The proof we are going to give is an application of RYLL-NARDZEWSKI's fixed point theorem [5] but some ideas go back to [7] (see also [4, XV. 6]). Notice that the Ryll-Nardzewski fixed point theorem has been already used in finite W^*-algebras in order to give a simpler proof for the existence of a finite trace [8].

2. Let $\mathcal{M}$ be a W^*-algebra and $\mathcal{G}$ a uniformly bounded multiplicative group in $\mathcal{M}$, i.e. such that

$$\|x\| \leqq M \quad (x \in \mathcal{G}), \tag{1}$$

where $M>0$ is independent of $x \in \mathcal{G}$.

Denote by $\overline{\mathcal{G}}^{co}$ the closure in the σ-topology (i.e. the weak topology induced in $\mathcal{M}$ by its predual $\mathcal{M}_*$ [6]) of the family

$$\left\{\sum_{j=1}^{n} \alpha_j x_j^* x_j \,\middle|\, x_j \in \mathcal{G},\ \alpha_j \geqq 0 \quad (j = 1, \dots, n), \quad \sum_{j=1}^{n} \alpha_j = 1\right\}. \tag{2}$$

It is obvious that $\overline{\mathcal{G}}^{co}$ is a convex set of positive elements in $\mathcal{M}$.

Lemma. *Let $\mathcal{G}$ be a uniformly bounded group in the W^*-algebra $\mathcal{M}$ and $M>0$ a bound for it. Then for every $a \in \overline{\mathcal{G}}^{co}$ we have*

$$M^{-2} \leqq a \leqq M^2. \tag{3}$$

In particular, a is invertible in $\mathcal{M}$ for any $a \in \overline{\mathcal{G}}^{co}$.

Proof. As $\mathcal{G}$ is a group it is easy to see that

(1′) $$M^{-2} \leqq x^* x \leqq M^2 \quad (x \in \mathcal{G}).$$

From this (3) follows immediately.

3. From now on we suppose that $\mathcal{M}$ is a finite W^*-algebra.

Theorem. *Let $\mathcal{G}$ be a uniformly bounded group in the finite W^*-algebra $\mathcal{M}$. Then there is a positive invertible element $b \in \mathcal{M}$, such that bxb^{-1} is unitary for any $x \in \mathcal{G}$.*

Proof. Let us define on $\mathcal{M}$ the mappings

(4) $$T_x(a) = x^* a x \quad (a \in \mathcal{M}),$$

where x runs over the group $\mathcal{G}$. It is easy to see that $\{T_x\}_{x \in \mathcal{G}}$ is a group of operators on $\mathcal{M}$ and $T_x(\overline{\mathcal{G}}^{co}) \subset \overline{\mathcal{G}}^{co}$, for every $x \in \mathcal{G}$.

Let a_1, a_2 be two arbitrary elements of $\overline{\mathcal{G}}^{co}$, $a_1 \neq a_2$. Since $\mathcal{M}$ is finite, there is a normal finite trace τ on $\mathcal{M}$ [6] such that $\tau((a_1 - a_2)^2) > 0$. As τ is a trace, we have for any $x \in \mathcal{G}$

$$\tau((a_1 - a_2)^2) = \tau(x^{-1}(a_1 - a_2) x^{*-1} x^* (a_1 - a_2) x) = \tau(y x^* (a_1 - a_2) x),$$

where $y = x^{-1}(a_1 - a_2) x^{*-1}$. By the previous Lemma we have $\|y\| \leqq 2M^4$, where $M > 0$ is a bound for $\mathcal{G}$.

From the Schwarz inequality we get

(5) $$\tau((a_1 - a_2)^2) \leqq 2M^4 \|\tau\|^{1/2} \tau((x^*(a_1 - a_2)x)^* x^* (a_1 - a_2) x)^{1/2}.$$

If we set

$$|x|_\tau = |\tau(x^* x)|^{1/2} \quad (x \in \mathcal{M}),$$

then by (5) we obtain

(6) $$\inf_{x \in \mathcal{G}} |T_x(a_1 - a_2)|_\tau > 0.$$

Let us consider the locally convex topology of $\mathcal{M}$ given by the family of seminorms $|x|_\varphi = (\varphi(x^* x))^{1/2}$, where φ runs over the set of all σ-continuous positive functionals on $\mathcal{M}$. This is the s-topology of $\mathcal{M}$ and it is stronger than the σ-topology. Moreover, a linear function on $\mathcal{M}$ is σ-continuous if and only if it is s-continuous [6, Corollary 1.8.10]. By the preceding Lemma, $\overline{\mathcal{G}}^{co}$ is σ-compact, i.e. it is compact in the weak topology of $\mathcal{M}$ corresponding to the s-topology. Then by (6), $\{T_x\}_{x \in \mathcal{G}}$ is a non-contracting group of linear s-continuous operators on $\overline{\mathcal{G}}^{co}$ in the sense of Ryll-Nardzewski [5]. By the Ryll-Nardzewski fixed point theorem [5], there is at least one $a \in \overline{\mathcal{G}}^{co}$ such that

(7) $$x^* a x = a \quad (x \in \mathcal{G}).$$

According to the previous Lemma, a is positive and invertible, therefore if $b=a^{1/2}$ then b^{-1} exists. Let us set

$$u_x = bxb^{-1} \quad (x \in \mathcal{G}). \tag{8}$$

Then we may write on account of (7)

$$u_x^* u_x = b^{-1}x^*bbxb^{-1} = 1.$$

Analogously, from (7) we have $xa^{-1}x^*=a^{-1}$; hence

$$u_x u_x^* = bxb^{-1}b^{-1}x^*b = 1,$$

consequently u_x is unitary for any $x \in \mathcal{G}$.

4. Let us recall some definitions from [1]. Suppose that $\mathcal{M}$ is a W^*-algebra of operators acting on a separable Hilbert space. Let Φ be a faithful normal positive linear mapping of $\mathcal{M}$ into itself, such that $\Phi^2=\Phi$.

A subalgebra $\mathcal{S}$ of $\mathcal{M}$ is said to be *subdiagonal* (with respect to Φ) if it has the following properties:

(i) $\mathcal{S}+\mathcal{S}^*$ is σ-dense in $\mathcal{M}$.

(ii) $\Phi(ab) = \Phi(a)\Phi(b) \quad (a, b \in \mathcal{S})$.

(iii) $\Phi(\mathcal{S}) \subset \mathcal{S} \cap \mathcal{S}^*$.

(iv) The nullspace of $(\mathcal{S} \cap \mathcal{S}^*)^2$ is trivial.

It is known that every subdiagonal subalgebra $\mathcal{S}$ of $\mathcal{M}$ is contained in a maximal subdiagonal subalgebra of $\mathcal{M}$ [1, Theorem 2.2.1].

Suppose now that $\mathcal{M}$ is finite.

A subdiagonal subalgebra $\mathcal{S}$ of $\mathcal{M}$ (with respect to Φ) is called *finite* if there is a faithful normal finite trace ϱ of $\mathcal{M}$ such that $\varrho(\Phi(x))=\varrho(x)$ for every $x \in \mathcal{M}$.

The next result is a generalization of Theorem 2.4 in [2].

Corollary. *Let $\mathcal{G}$ be a uniformly bounded group in a finite W^*-algebra $\mathcal{M}$ of operators acting on a separable Hilbert space. Let $\mathcal{S}$ be a finite maximal subdiagonal subalgebra in $\mathcal{M}$. Then there is an invertible element $a \in \mathcal{S}$ such that $a^{-1} \in \mathcal{S}$ and axa^{-1} is unitary for each $x \in \mathcal{G}$.*

Proof. According to the previous Theorem, there is a positive invertible element $b \in \mathcal{M}$, such that bxb^{-1} is unitary for every $x \in \mathcal{G}$. On account of [1, Theorem 4.2.1], $b=ua$, where u is unitary and $a \in \mathcal{S} \cap \mathcal{S}^{-1}$. Now, it is obvious that axa^{-1} is unitary when $x \in \mathcal{G}$.

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