On a problem of Kátai

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1. In his paper [1] I. KÁTAI set the following problem: Let f and g be two additive arithmetical functions; we suppose there exists a $l\in\mathbf{C}$ such that

$$\operatorname*{Lim}_{n\to+\infty}\{g(2n+1)-f(n)\}=l;$$

what can we deduce for the form of f or g?

The purpose of our paper is to prove the two following results:

(1) *Let g be an additive function; if f is a completely additive function, and if there exists an $l\in\mathbf{C}$ such that*

$$\lim_{n\to+\infty}\{g(2n+1)-f(n)\}=l \quad (\textit{hypothesis } \mathrm{H}),$$

then $f(n)=\dfrac{l}{\log 2}\log n$, and for every odd prime p and every positive integer d we have $g(p^\alpha)=f(p^\alpha)$.

(2) *Let g be an additive function; if f is a completely additive function and if there exists an $M\in\mathbf{R}^+$ such that*

$$|g(2m+1)-f(m)|\leqq M \quad \textit{for every} \quad m\in\mathbf{N}^*,$$

then $f(n)=C\log n$, where C is a constant.

1.1. Remarks. 1) It follows from the conclusion of (1) that $g(n)=f(n)$ for every odd $n\in\mathbf{N}^*$; of course, we cannot deduce anything about even n's, because only values of g on odd integers are involved in the hypothesis. 2) From the conclusion of (2), we can easily deduce that $f(2n+1)-g(2n+1)$ is bounded independently of n.

Let us verify assertion 2):

There is an A such that $|h(m)|\leqq A$; furthermore, we have $|g(2m+1)-h(m)-C\log m|\leqq M$ Then, it follows that $|g(2m+1)-C\log m|\leqq M+A$.

But there exists a B such that: $|C[\log(2m+1)-\log m]|\leqq B$. Then we have $|g(2m+1)-C\log(2m+1)|\leqq A+B+M$; and therefore

$$|g(2m+1)-\{C\log(2m+1)+h(2m+1)\}|\leqq 2A+B+M,$$

i.e.: $|g(2m+1)-f(2m+1)|\leqq 2A+M+B$.

2. Proof of (1). 2.1. First, we have $g(2n+1)-f(n)-l=o(1)$ and $g(2n-1)-f(n-1)-l=o(1)$ $(n\to+\infty)$; since $(2n+1, 2n-1)=1$, we also have $g(2n+1)+g(2n-1)=g(4n^2-1)$; hence

(A) $$g(4n^2-1)-f(n)-f(n-1)-2l=o(1)\quad(n\to+\infty).$$

Moreover, $g(4n^2-1)=g[2(2n^2-1)+1]$, and it follows from hypothesis (H) that:

(B) $$g(4n^2-1)-f(2n^2-1)-l=o(1)\quad(n\to+\infty).$$

We deduce from (A) and (B) that

(C) $$f(2n^2-1)-f(n)-f(n-1)-l=o(1)\quad(n\to+\infty).$$

2.2. Using hypothesis (H) we get:

$$g[(2n+1)^2]-f(2n(n+1))-l=o(1)\quad\text{and}\quad g[(2n-1)^2]-f(2n(n-1))-l=o(1)$$
$$(n\to+\infty).$$

But $(2n+1, 2n-1)=1$; hence $g[(2n+1)^2]+g[(2n-1)^2]=g[(4n^2-1)^2]$ and it follows that

(A′) $$g[(4n^2-1)^2]-f(2n(n-1))-f(2n(n+1))-2l=o(1)\quad(n\to+\infty).$$

Now we notice that $g[(4n^2-1)^2]=g[8n^2(2n^2-1)+1]$; using hypothesis (H) we get

(B′) $$g[(4n^2-1)^2]-f[4n^2(2n^2-1)]-l=o(1)\quad(n\to+\infty).$$

This, together with (A′), yields

$$f[4n^2\times(2n^2-1)]-f[2n(n+1)]-f[2n(n-1)]-l=o(1)\quad(n\to+\infty).$$

Since f is completely additive, we get

(C′) $$f(2n^2-1)-f(n-1)-f(n+1)-l=o(1)\quad(n\to+\infty).$$

2.3. We now replace $f(2n^2-1)$ in (C′) by its value obtained from (C); thus, we have $f(n+1)-f(n)=o(1)$ $(n\to+\infty)$. By a well-know theorem of Erdős ([2]), we have $f(n)=C\log n$, where C is a constant. Thus, (C) becomes:

$$\lim_{n\to+\infty}\{C\times(\log(2n^2-1)-\log n-\log(n-1))-l\}=0,$$

which implies $C=\frac{l}{\log 2}$.

Now, hypothesis (H) becomes:

$$g(2n+1)-\frac{l}{\log 2}\log n-l=o(1),\quad \text{i.e.}\quad g(2n+1)-\frac{l}{\log 2}\log 2n=o(1)\quad (n\to+\infty).$$

But $\log(2n+1)-\log 2n=o(1)$ $(n\to+\infty)$. Therefore:

$$\text{(D).}\qquad g(2n+1)-\frac{l}{\log 2}\log(2n+1)=o(1)\quad (n\to+\infty).$$

Now let $\alpha\in\mathbf{N}^*$ and let p be any odd prime; we take in (D) $2n+1=p^\alpha\times(2pm+1)$. We thus get

$$g(p^\alpha)+g(2pm+1)-\frac{l}{\log 2}\log p^\alpha-\frac{l}{\log 2}\log(2pm+1)=o(1)\quad (m\to+\infty).$$

But by (D) we have

$$g(2pm+1)-\frac{l}{\log 2}\log(2pm+1)=o(1)\quad (m\to+\infty).$$

It follows that

$$g(p^\alpha)-\frac{l}{\log 2}\log p^\alpha=o(1)\quad (m\to+\infty),$$

which implies

$$g(p^\alpha)=\frac{l}{\log 2}\log p^\alpha.$$

3. Proof of (2). Using the same method as for the proof of (1), we obtain:

$$\text{I.}\qquad |f(2n^2-1)-f(n)-f(n-1)|\leqq 3M.$$

$$\text{II.}\qquad |f(2n^2-1)-f(n-1)-f(n+1)|\leqq 3M.$$

$$\text{III.}\qquad |f(n+1)-f(n)|\leqq 6M.$$

By a result of Wirsing ([3]), we obtain: $f(n)=C\log n+h(n)$, where h is a bounded additive function. Since h is completely additive (because f is completely additive), h is identically zero.

References

[1] I. Kátai, Some results and problems in the theory of additive functions, *Acta Sci. Math.,* **30** (1969), 305—312.

[2] P. Erdős, On the distribution function of additive functions, *Ann. of Math.,* **47** (1946), 4—20.

[3] E. Wirsing, A characterization of the logarithm as an additive function, *Proc. Rome Conference of Number Theory,* 1968.

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