On a property of operators of class C_0

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In their paper [1] CLANCEY and MOORE prove (as a step to their main result) that for any contraction T of class C_0 and with finite defect indices there exists a nonzero compact operator commuting with T.

Recall that T is of class C_0 if it is completely non-unitary and $\varphi(T)=0$ for some inner function φ; among these functions there is a minimal one (i.e. which is a divisor in H^∞ of all the others), denoted by m_T. To every given nonconstant inner function m there exist contractions T of class C_0 with m_T equal to m; the simplest example is the operator $T=S(m)$ on the function space $\mathfrak{H}(m)$, defined by

$$\mathfrak{H}(m) = H^2 \ominus mH^2, \quad S(m)u = P_{\mathfrak{H}(m)}(\chi u) \quad \text{for} \quad u \in \mathfrak{H}(m), \tag{1}$$

where $\chi(e^{it})=e^{it}$ and H^2 is the Hardy—Hilbert space for the unit disc. See [3], Chapter III.

By theorems of SARASON [2] the operators Φ commuting with $S(m)$ are precisely those which can be written in the form

$$\Phi u = P_{\mathfrak{H}(m)}(\varphi u) \quad (u \in \mathfrak{H}(m)), \tag{2}$$

where φ is any fixed function in H^∞. Moreover, Φ is compact if and only if φ/m is, on the unit circle, the sum of a continuous function and of an H^∞ function. From (1) and (2) it follows, finally, that $\Phi \neq 0$ if and only if $\varphi \notin mH^2$, i.e. if $\varphi/m \notin H^\infty$.

Now for every nonconstant inner m there exists even $\varphi \in H^\infty$ such that φ/m is continuous on the unit circle, but not belonging to H^∞. If m has at least one (simple) Blaschke factor b then an obvious choice is $\varphi=m/b$. If m is a purely singular inner function, such a φ was constructed in [1].

Thus every operator $S(m)$ has a nonzero compact operator in its commutant. This property is shared by all contractions of class C_0. Indeed, we have

Theorem. *For every contraction T of class C_0 on a Hilbert space $\mathfrak{H}$ there exists a nonzero compact operator commuting with T.*

Proof. By virtue of Proposition 2 in [4] we have $T \succ S(m) \oplus T_1$ [1]) for some contraction T_1 of class C_0 and for $m=m_T$. Applying this to T^* as well and taking adjoints it also follows that

$$S(m)\oplus T_2 \succ T \succ S(m)\oplus T_1 \tag{3}$$

with some contractions T_i on spaces $\mathfrak{H}_i$ ($i=1, 2$), and with $m=m_T$. Hence there exist quasi-affinities

$$X_1 : \mathfrak{H}(m)\oplus \mathfrak{H}_1 \to \mathfrak{H}, \quad X_2 : \mathfrak{H} \to \mathfrak{H}(m)\oplus \mathfrak{H}_2$$

such that

$$TX_1 = X_1(S(m)\oplus T_1), \quad (S(m)\oplus T_2)X_2 = X_2 T. \tag{4}$$

Now choose a nonzero compact operator Φ commuting with $S(m)$ and define, for $h\in\mathfrak{H}$,

$$Fh = X_1(\Phi P_2 X_2 h\oplus 0_1), \tag{5}$$

where 0_1 denotes the zero vector in $\mathfrak{H}_1$ and P_2 is the orthogonal projection of $\mathfrak{H}(m)\oplus\mathfrak{H}_2$ onto its subspace $\mathfrak{H}(m)\oplus\{0\}$, which we identify with $\mathfrak{H}(m)$.

Clearly, $P_2(S(m)\oplus T_2)=S(m)P_2$ and by (4) we have for $h\in\mathfrak{H}$

$$\begin{aligned} FTh &= X_1(\Phi P_2 X_2 Th\oplus 0_1) = X_1(\Phi P_2(S(m)\oplus T_2)X_2 h\oplus 0_1) = \\ &= X_1(\Phi\, S(m)P_2 X_2 h\oplus 0_1) = X_1(S(m)\Phi P_2 X_2 h\oplus 0_1) = \\ &= X_1(S(m)\oplus T_1)(\Phi P_2 X_2 h\oplus 0_1) = TX_1(\Phi P_2 X_2 h\oplus 0_1) = TFh. \end{aligned}$$

Hence, T commutes with F. Since Φ is compact so is F by its definition (5). Moreover $F\neq 0$. For, $F=0$ implies $\Phi P_2 X_2=0$ because X_1 has zero kernel, $\Phi P_2 X_2=0$ implies $\Phi P_2=0$ because X_2 has dense range, and $\Phi P_2=0$ simply means $P_2=0$. This contradicts the fact that $\mathfrak{H}(m)\neq\{0\}$ for nonconstant inner m.

Thus F is a nonzero compact operator on $\mathfrak{H}$ commuting with T.

Remark. F is, in general, not included in the weakly closed algebra generated by T and $I_{\mathfrak{H}}$. Hence, we have no immediate generalization of the Theorem in [1].

References

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[2] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.

[3] B. Sz.-Nagy—C. Foiaş, *Harmonic analysis of operators on Hilbert space,* Akadémiai Kiadó (Budapest, 1970).

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[1]) $A\succ B$ means that there exists a „quasi-affinity" X (i.e. an operator with zero kernel and dense range) such that $AX=XB$.