On functions of bounded deviation

By DANIEL WATERMAN in Syracuse (N. Y., U.S.A.)

1. Let f be a real or complex valued function of a real variable with period 2π and integrable over $[0, 2\pi]$. Let χ_I denote the characteristic function of an interval $I \subset [0, 2\pi]$. The function f was said by HADAMARD [3], who restricted himself to continuous functions, to be of *bounded deviation* (écart fini) if for some positive $C < \infty$ the modulus of the n-th Fourier coefficient of $f\chi_I$ is less than $C/|n|$ for every n and I. Hadamard observed that continuous functions of bounded variation have this property (as, indeed, do all BV functions) and Hille, Bray, and Alexits gave numerous examples of functions not of bounded variation but of bounded deviation (for detailed references see ZYGMUND [4, p. 229]).

We shall consider the effect on these functions of a change of variable, i.e., we consider the functions $f \circ g$ where g is a homeomorphism of $[0, 2\pi]$ onto itself. We shall show that $f \circ g$ is of bounded deviation if and only if f is equivalent, in a certain sense, to a function of bounded variation. If we were to assume that f is continuous or regulated (i.e., that its discontinuities are simple), then the need for the equivalence relation in this result would vanish.

A set $E \subset [0, 2\pi]$ is said to have *universal measure zero* (UMZ) if for every homeomorphism g of $[0, 2\pi]$ with itself (i.e., every change of variable), we have that $g(E)$ is Lebesgue measurable and

$$m(g(E)) = 0.$$

Two functions will be said to be *equivalent* if they are equal except on a UMZ-set.

Our principal result is the following.

Theorem. *A function is of bounded deviation for every change of variable if and only if it is equivalent to a function of bounded variation.*

The next section of this paper is concerned with some preliminary results. In § 3, we prove our theorem for the special case of regulated functions and, in § 4, we prove the general result. In the arguments of § 3 and 4 we can assume, without loss of generality, that f is real valued.

2. A function f is said to be *universally essentially bounded* if there is an $M<\infty$ such that $\{|f|>M\}$ is a UMZ-set. For real valued functions, upper and lower universal essential bounds may be defined in the obvious manner.

We shall use the following lemmas:

Lemma 1. *If f is of bounded deviation, then f is essentially bounded.*

Lemma 2. *A function f is universally essentially bounded if and only if $f\circ g$ is essentially bounded for every change of variable g.*

It follows from these results that the functions we consider may be assumed to be universally essentially bounded since, for each change of variable g, $f\circ g$ will be of bounded deviation and, therefore, essentially bounded.

Actually neither of these lemmas is required to obtain this fact, although we believe them to be of independent interest. In order for $f\circ g$ to be of bounded deviation for each g, it must be integrable, hence measurable, for each g. As indicated in our paper [2] with GOFFMAN, if we suppose, as we may, that f is real, this implies that for each real k, $\{f>k\}$ is either of universal measure zero or contains a perfect set. Thus if f were not universally essentially bounded, there would exist a change of variable g such that $f\circ g$ would not be integrable.

As the proof of lemma 1 will show, the hypothesis may be considerably weakened. Actually, we need only consider a subsequence of the Fourier coefficients of $\chi_I f$. The lemma, as stated above, was proven by CAVENY [1], but we shall give a much simpler demonstration than his.

We shall show that f is bounded on its Lebesgue set. Suppose x is a Lebesgue point of f. Then for any positive integer n there is an integer $k\in[0, 2n-1]$ so that $x\in\left[\frac{k}{n}\pi, \frac{k+1}{n}\pi\right]$. The hypothesis implies that there is an $M<\infty$ independent of n such that

$$M>\left|\frac{n}{\pi}\int_{k\pi/n}^{(k+1)\pi/n} f(t)\sin nt\,dt\right| \geqq \left|\frac{n}{\pi}\left|\int_{k\pi/n}^{(k+1)\pi/n} f(x)\sin nt\,dt\right| - \frac{n}{\pi}\int_{k\pi/n}^{(k+1)\pi/n} |f(x)-f(t)|\,dt = \right.$$

$$=\frac{2}{\pi}|f(x)|-o(1)$$

as $n\to\infty$. Thus Lemma 1 is established.

Now in one direction, Lemma 2 is obvious. We need only show that if f is not universally essentially bounded, then there is a change of variable g such that $f\circ g$ is not essentially bounded. Without loss of generality, we may assume that the restriction of f to any right neighborhood of 0 is not universally bounded. Then there is an increasing homeomorphism g_1 of $[0, 2\pi]$ onto itself and $a_1 \in (0, \pi)$ such that

$$m(\{|f \circ g| > 1\} \cap (a_1, 2\pi)) > 0.$$

There is a g_2, an increasing homeomorphism of $[0, a_1]$ onto $[0, g_1(a_1)]$, such that for some $a_2 \in (0, a_1/2)$, with $g_2(a_2) < g_1(a_1)/2$,

$$m(\{|f \circ g_2| > 2\} \cap (a_2, a_1)) > 0.$$

Continuing in this manner, we construct g_n and a_n, $n=2, 3, \ldots$, such that g_n is an increasing homeomorphism of $[0, a_{n-1}]$ onto $[0, g_{n-1}(a_{n-1})]$, $a_n \in (0, a_{n-1}/2)$, $g_n(a_n) < g_{n-1}(a_{n-1})/2$, and

$$m(\{|f \circ g_n| > n\} \cap (a_n, a_{n-1})) > 0.$$

Let g be the increasing function equal to g_1 on $[a_1, 2\pi]$, to g_n on $[a_n, a_{n-1}]$, $n=2, 3, \ldots$, and to 0 at 0. Then g is a change of variable with the property that

$$m(\{|f \circ g| > n\}) > 0$$

for all n.

3. We turn now to the proof of our theorem. We shall assume, at first, that f is equivalent to a regulated function. Identifying f with that function, we have that $f(x+)$ and $f(x-)$ exist for each x and $f(x) = \frac{1}{2}(f(x+) + f(x-))$. If f is not of bounded variation, then without loss of generality we may assume that there exist ponits of continuity f, a_{ni}, b_{ni}, $i \leqq k_n$, $a_{n1} \searrow 0$ as $n \to \infty$, such that

$$0 < \ldots < a_{n1} < b_{n1} < a_{n2} < b_{n2} < \ldots < b_{nk_n} < a_{n-1,1} < \ldots < 2\pi$$

and

$$v_n = \sum_{1}^{k_n} (f(b_{ni}) - f(a_{ni})) \to \infty$$

as $n \to \infty$. Note that this implies $k_n \to \infty$. Choose a positive integer m_1 so that $k_1/m_1 < 1$ and, for $n = 1, 2, 3, \ldots$, choose integers m_{n+1} so that

$$m_{n+1} > (2k_{n+1} + 1) m_n.$$

Consider the intervals

$$\left[\frac{1}{m_n}\pi, \frac{2}{m_n}\pi\right], \left[\frac{2}{m_n}\pi, \frac{3}{m_n}\pi\right], \ldots, \left[\frac{2k_n}{m_n}\pi, \frac{2k_n+1}{m_n}\pi\right].$$

Let g assume the value a_{n1} at the center of the first interval, b_{n1} at the center of the second interval, a_{n2} at the center of the third interval, and so on. Suppose we exclude from each of these intervals the portion contained in intervals centered at

$\frac{2}{m_n}\pi, \frac{3}{m_n}\pi, \ldots, \frac{2k_n}{m_n}\pi$ whose lengths are small compared to π/m_n. On each of the remaining intervals, let g be linear and of very small slope. On each of the small excluded intervals, let g be linear and so that, on $\left[\frac{1}{m_n}\pi, \frac{2k_n+1}{m_n}\pi\right]$, g is continuous. If the almost horizontal portions of g are sufficiently flat, we see that we can set $g(0)=0$, $g(2\pi)=2\pi$, and define g to be linear on each component of the complement of the intervals used and continuous and strictly increasing on $[0, 2\pi]$. Now

$$\int_{\pi/m_n}^{(2k_n+1)\pi/m_n} f\circ g(x)\sin m_n x\,dx = \sum_{i=0}^{2k_n-1}\int_{\pi/m_n}^{2\pi/m_n} f\circ g(x+i\pi/m_n)\sin m_n(x+i\pi/m_n)\,dx =$$

$$= \sum_{i=1}^{k_n}\int_0^{\pi/m_n}\left[f\circ g(x+2i\pi/m_n) - f\cdot g(x+(2i-1)\pi/m_n)\right]\sin m_n x\,dx = \frac{2}{m_n}(v_n+h_n)$$

where $h_n=o(1)$ as $n\to\infty$ if the almost horizontal segments of g are of sufficiently small slope and the small intervals selected about $\frac{2}{m_n}\pi, \ldots, \frac{2k_n}{m_n}\pi$ are sufficiently small. Hence

$$m_n\int_{\pi/m_n}^{(2k_n+1)\pi/m_n} f\circ g(x)\sin m_n x\,dx \to \infty \quad \text{as} \quad n\to\infty.$$

4. If we now assume that f is not equivalent to a regulated function, but is universally essentially bounded, then, as we have shown in § 3 of our paper [2] with Goffman, we may assume that the sets $\{f>1\}\cap(0,\delta)$ and $\{f<-1\}\cap(0,\delta)$ are not UMZ-sets for every $\delta>0$ and, therefore, that there is a sequence $\{a_n\}$, $a_0=2\pi$, $a_n\searrow 0$, such that for $n=0, 1, 2, \ldots$,

$$\{(-1)^n f>1\}\cap(a_{n+1}, a_n)$$

is not a UMZ-set.

We could proceed to give a direct argument based on the above to show that there is a change of variable g such that $f\circ g$ is not of bounded deviation. It is more economical, however, to pattern our argument after that of the previous section.

For each $n=0, 1, 2, \ldots$, there is an increasing homeomorphism h_n of $[a_{n+1}, a_n]$ onto itself such that

$$m(\{(-1)^n f\circ h_n>1\}\cap(a_{n+1}, a_n))>0.$$

Let h be the change of variable on $[0, 2\pi]$ whose restriction to $[a_{n+1}, a_n]$ is h_n for each n. Let $F=f\circ h$. Then, proceeding as in the previous section, choose a_{ni}, b_{ni} such that $F(b_{ni})>1$, $F(a_{ni})<-1$, $k_n\to\infty$ and a_{ni} and b_{ni} are points of *approximate continuity* of F. Let M be the universal essential upper bound of $|f|$. Proceed as before to define g. If the small intervals chosen about $\frac{2}{m_n}\pi, \frac{3}{m_n}\pi, \ldots, \frac{2k_n}{m_n}\pi$ on which g

rises abruptly have length $\varepsilon_n = o(1/m_n k_n)$, then, letting I_n denote the union of these intervals, we have

$$\left| \int_{I_n} F \circ g(x) \sin m_n x \, dx \right| < 2k_n M \varepsilon_n = o(1/m_n).$$

On the almost horizontal portions, g can be chosen so flat that the *relative* measure of the set $\{(-1)^i F \circ g \geqq 1\}$ in $\left(\frac{i}{m_n}\pi, \frac{i+1}{m_n}\pi\right) \setminus I_n$ is as close to one as we wish, say $> 1-\delta_n$ with $\delta_n = o\left(\frac{1}{k_n}\right)$. Then

$$\left| \int_{\pi/m_n}^{(2k_n+1)\pi/m_n} F \circ g(x) \sin m_n x \, dx \right| = \left| \int_{(\pi/m_n,\, (2k_n+1)\pi/m_n) \setminus I_n} + \int_{I_n} \right| \geqq$$

$$\geqq 4k_n/m_n - 2k_n M \delta_n \pi/m_n + o(1/m_n) = 4k_n/m_n + o(1/m_n)$$

and so $f \circ (h \circ g)$ is not of bounded deviation.

References

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SYRACUSE UNIVERSITY
SYRACUSE, NEW YORK 13 210
U.S.A.

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